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Empirical Sets

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The pillar concept of Foulis and Randall's school is surely that of a manual of operations. They chose to regard an operation as the set of possible outcomes, thereby taking a manual of operations to be a family of partially overlapping operations. Our previous work is a development of their ideas in two points. First, each operation is represented not by the set of possible outcomes, but by the complete Boolean algebra of observable events. Second, since each complete Boolean algebra **B** possesses the Scott–Solovay model $V^{(\mathbf{B})}$ of classical set theory as its higher-order companion, the Scott–Solovay universes of all the operations in the manual lump together into a family of Boolean set theories interconnected by geometric morphisms, which we suggestively designated an empirical set theory. The principal concern of this paper is to show how to get a cross-operational set concept by choosing an internal set within $V^{(\mathbf{B})}$ for each operation **B** in the manual and bundling them up. The resulting structure is denominated an empirical set. We show that the category of empirical sets is complete, is cocomplete, has a subobject classifier for well-rounded subobjects, and has exponentials only for degraded objects.

INTRODUCTION

Foulis and Randall (1972; Randall and Foulis, 1973) discussed manuals of operations so as to formalize the operational and epistemological aspects of empirical sciences ranging from physics and biology to sociology and artificial intelligence. In their literature an operation is identified with a set of possible outcomes and a manual of operations is thought to be a family of partially overlapping operations. Although each operation enjoys classical logic and classical statistics, the logic and the statistics of a manual of operations as a whole are not classical in general, since the existence of a grand operation refining all the operations in the manual is rather an exception than a rule.

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In a previous paper (Nishimura, 1993*b*), as a maverick of Foulis and Randall's school, we modernized their ideas by using category theory and outlined a higher-order generalization. There each operation is represented by a complete Boolean algebra **B**, and our empirical set theory is a family of Scott–Solovay universes $V^{(\mathbf{B})}$ interconnected by geometric morphisms. Our ambition is modest enough. We aim ultimately to do the same thing to so-called quantum logic as Grothendieck did to algebraic geometry some decades ago.

There we showed that so long as the manual of operations is well behaved, the real numbers of our empirical set theory can be identified externally with the observables on its logic. Now a question naturally occurs to us. What conceptual status does the family of the internal sets of real numbers within $V^{(\mathbf{B})}$ for all the operations **B** in the manual occupy internally? A similar question occurs for truth-value objects. What role does the internal truth-value objects within $V^{(\mathbf{B})}$ for all the operations **B** in the manual play?

The principal objective of this study is to introduce a cross-operational notion of an empirical set, which is hopefully an answer to the above questions, and then to discuss the fundamental properties of the category of empirical sets. It is shown that the category is complete, is cocomplete, has a subobject classifier for well-turned subobjects, and has exponentials for degenerate empirical sets. These are the topics of Section 2. Section 1 is devoted to preliminary considerations on Boolean set theory. We have made every effort to render the paper as accessible as possible. As for category theory and topos theory in particular, familiarity with MacLane (1971) up to Chapter V and an elementary textbook on topos theory such as Goldblatt (1979) should be more than sufficient. Even the definition of a geometric morphism is not a prerequisite. Since the details of the construction of $V^{(\mathbf{B})}$ are appreciably tedious, we choose to use the category $\mathfrak{BSh}(\mathbf{B})$ of sheaves over **B** instead. We prefer to do everything as concretely as possible rather than pursue full sophistication. The rest of this section is devoted to preliminaries on orthogonal categories, manuals, and Boolean locales, in that order. For the details of orthogonal categories and manuals of Boolean locales the reader is referred respectively to Nishimura (1995*a*) and Nishimura (1993*b*), though familiarity with them is not obligatory.

Orthogonal Categories

A pair $(\mathfrak{K}, \mathfrak{os}_{\mathfrak{K}})$ of a category $\mathfrak{K}$ and a class $\mathfrak{os}_{\mathfrak{K}}$ of diagrams in $\mathfrak{K}$ is called an *orthogonal category* if it satisfies the following conditions:

(1) The category $\mathfrak{K}$ has an initial object.
(2) Every diagram in $\mathfrak{os}_{\mathfrak{K}}$ is of the form $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda}$.

(3) For any small family $\{\mathbf{X}\}_{\lambda \in \Lambda}$ of objects in $\mathfrak{K}$ there exist an object $\mathbf{Y}$ in $\mathfrak{K}$ and a family $\{\mathbf{f}_\lambda\}_{\lambda \in \Lambda}$ of morphisms $\mathbf{f}_\lambda$: $\mathbf{X}_\lambda \to \mathbf{Y}$ in $\mathfrak{K}$ such that the diagram $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$.

(4) Given a small family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of objects in $\mathfrak{K}$, if diagrams $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda}$ and $\{\mathbf{X}_\lambda \overset{\mathbf{g}_\lambda}{\to} \mathbf{Z}\}_{\lambda \in \Lambda}$ lie in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$, then there exists a unique morphism $\mathbf{h}$: $\mathbf{Y} \to \mathbf{Z}$ in $\mathfrak{K}$ such that $\mathbf{g}_\lambda = \mathbf{h} \circ \mathbf{f}_\lambda$ for each $\lambda \in \Lambda$.

(5) Given diagrams $\{\mathbf{Y}_\lambda \overset{\mathbf{g}_\lambda}{\to} \mathbf{Z}\}_{\lambda \in \Lambda}$ and $\{\mathbf{X}_\delta \overset{\mathbf{f}_\delta}{\to} \mathbf{Y}_\lambda\}_{\delta \in \Delta_\lambda}$ $(\lambda \in \Lambda)$ in $\mathfrak{K}$, the diagram $\{\mathbf{X}_\delta \xrightarrow{\mathbf{g}_\lambda \circ \mathbf{f}_\delta} \mathbf{Z} \mid \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$ iff all the diagrams $\{\mathbf{Y}_\lambda \overset{\mathbf{g}_\lambda}{\to} \mathbf{Z}\}_{\lambda \in \Lambda}$ and $\{\mathbf{X}_\delta \overset{\mathbf{f}_\delta}{\to} \mathbf{Y}_\lambda\}_{\delta \in \Delta_\lambda}$ $(\lambda \in \Lambda)$ lie in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$, where the sets Δ_λ are assumed to be mutually disjoint.

(6) If a diagram $\{\mathbf{X}_\delta \overset{\mathbf{f}_\delta}{\to} \mathbf{Y} \mid \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$, then there exist diagrams $\{\mathbf{X}_\delta \overset{\mathbf{g}_\delta}{\to} \mathbf{Z}_\lambda\}_{\delta \in \Delta_\lambda}$ $(\lambda \in \Lambda)$ and $\{\mathbf{Z}_\lambda \overset{\mathbf{h}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda}$ such that $\mathbf{f}_\delta = \mathbf{h}_\lambda \circ \mathbf{g}_\delta$ for any $\lambda \in \Lambda$ and any $\delta \in \Delta_\lambda$, where the sets Δ_λ are assumed to be mutually disjoint.

(7) If $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda}$ is a diagram in $\mathfrak{K}$ and $\{\mathbf{Z}_\delta \overset{\mathbf{g}_\delta}{\to} \mathbf{Y}\}_{\delta \in \Delta}$ is also a diagram in $\mathfrak{K}$ with $\mathbf{Z}_\delta$ being an initial object of $\mathfrak{K}$ for each $\delta \in \Delta$, then the diagram $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda}$ is in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$ iff the diagram $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda} \cup \{\mathbf{Z}_\delta \overset{\mathbf{g}_\delta}{\to} \mathbf{Y}\}_{\delta \in \Delta}$ is in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$.

(8) If $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ is an isomorphism in $\mathfrak{K}$, then the diagram $\{\mathbf{X} \overset{\mathbf{f}}{\to} \mathbf{Y}\}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$.

(9) Given a diagram $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda}$ in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$, if $\mathbf{f}_{\lambda_1}$ and $\mathbf{f}_{\lambda_2}$ happen to be the same morphism for some distinct $\lambda_1, \lambda_2 \in \Lambda$ (so that $\mathbf{X}_{\lambda_1} = \mathbf{X}_{\lambda_2}$), then $\mathbf{X}_{\lambda_1} = \mathbf{X}_{\lambda_2}$ is an initial object of $\mathfrak{K}$.

(10) If a diagram $\{\mathbf{X} \overset{\mathbf{f}}{\to} \mathbf{Y}\}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$, then $\mathbf{f}$ is an isomorphism.

(11) Given diagrams $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda}$ and $\{\mathbf{X}_\delta \overset{\mathbf{g}_\delta}{\to} \mathbf{Y}\}_{\delta \in \Delta}$ in $\mathfrak{K}$, if both the diagram $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda}$ and the diagram $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda} \cup \{\mathbf{X}_\delta \overset{\mathbf{g}_\delta}{\to} \mathbf{Y}\}_{\delta \in \Delta}$ are in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$, then $\mathbf{X}_\delta$ is an initial object for each $\delta \in \Delta$.

Unless confusion may arise, the category $\mathfrak{K}$ itself is called an *orthogonal category* by abuse of language. A diagram $\{\mathbf{X}_\lambda \overset{\mathbf{f}_\lambda}{\to} \mathbf{Y}\}_{\lambda \in \Lambda}$ in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$ is called an *orthogonal sum diagram*, in which $\mathbf{Y}$ is called an *orthogonal sum* of $\mathbf{X}_\lambda$'s and is denoted by $\Sigma_{\lambda \in \Lambda} \oplus \mathbf{X}_\lambda$. Thus the class $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$ is the class of orthogonal sum diagrams in $\mathfrak{K}$. A morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ is called an *embedding* if there exists a morphism $\mathbf{g}$: $\mathbf{Z} \to \mathbf{Y}$ in $\mathfrak{K}$ such that the diagram $\mathbf{X} \overset{\mathbf{f}}{\to} \mathbf{Y} \overset{\mathbf{g}}{\leftarrow} \mathbf{Z}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$. Two embeddings $\mathbf{f}$: $\mathbf{Y} \to \mathbf{X}$ and $\mathbf{g}$: $\mathbf{Z} \to \mathbf{X}$ with the same codomain are said to be *equivalent* if there exists an isomorphism $\mathbf{h}$: $\mathbf{Y} \to \mathbf{Z}$ in $\mathfrak{K}$ such that $\mathbf{f} = \mathbf{g} \circ \mathbf{h}$. An object in $\mathfrak{K}$ is called *trivial* if it is an initial object of $\mathfrak{K}$. A trivial object of $\mathfrak{K}$ can be regarded as the orthogonal sum of the empty family of objects in $\mathfrak{K}$.

Manuals

Let $\mathfrak{K}$ be an orthogonal category and $\mathfrak{M}$ a subcategory of it. A diagram in $\mathfrak{K}$ is said to be *in* $\mathfrak{M}$ if all the objects and morphisms occurring in the diagram lie in $\mathfrak{M}$. Objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathfrak{M}$ are said to be *$\mathfrak{M}$-orthogonal*, in notation $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, if there exists an orthogonal sum diagram $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Z} \xleftarrow{\mathbf{g}} \mathbf{Y}$ of $\mathfrak{K}$ lying in $\mathfrak{M}$. An object of $\mathfrak{M}$ is said to be *$\mathfrak{M}$-trivial* if it is a trivial object of $\mathfrak{K}$ and also an initial object of $\mathfrak{M}$. An object $\mathbf{X}$ of $\mathfrak{M}$ is said to be *$\mathfrak{M}$-maximal* if for any object $\mathbf{Y}$ of $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ implies that $\mathbf{Y}$ is $\mathfrak{M}$-trivial. Objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathfrak{M}$ are said to be *$\mathfrak{M}$-equivalent*, in notation $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$, provided that for any objects $\mathbf{Z}$ of $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ iff $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$. Obviously $\mathfrak{M}$-equivalence is an equivalence relation among the objects of $\mathfrak{M}$. We denote by $[\mathbf{X}]_{\mathfrak{M}}$ the equivalence class of an object $\mathbf{X}$ of $\mathfrak{M}$ with respect to $\mathfrak{M}$-equivalence. An orthogonal sum diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{X}\}_{\lambda \in \Lambda}$ of $\mathfrak{K}$ lying in $\mathfrak{M}$ is said to be an *orthogonal $\mathfrak{M}$-sum diagram* if for any orthogonal sum diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}'_\lambda} \mathbf{X}'\}_{\lambda \in \Lambda}$ of $\mathfrak{K}$ lying in $\mathfrak{M}$ the unique morphism $\mathbf{g}$: $\mathbf{X} \to \mathbf{X}'$ of $\mathfrak{K}$ with $\mathbf{g} \circ \mathbf{f}_\lambda = \mathbf{f}'_\lambda$ for any $\lambda \in \Lambda$ belongs to $\mathfrak{M}$, in which $\mathbf{X}$ is called an *orthogonal $\mathfrak{M}$-sum* of $\mathbf{X}_\lambda$'s and is denoted by $\Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$. If Λ is a finite set, say, $\Lambda = \{1, 2\}$, then such a notation as $\mathbf{X}_1 \oplus_{\mathfrak{M}} \mathbf{X}_2$ is preferred. Note that an $\mathfrak{M}$-trivial object of $\mathfrak{M}$, if it exists, can be regarded as an orthogonal $\mathfrak{M}$-sum of the empty family of objects of $\mathfrak{M}$. A morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ is called an *$\mathfrak{M}$-embedding* if there exists a morphism $\mathbf{g}$: $\mathbf{Z} \to \mathbf{Y}$ such that the diagram $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xleftarrow{\mathbf{g}} \mathbf{Z}$ is an orthogonal $\mathfrak{M}$-sum diagram. Given objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathfrak{M}$, if there exists an $\mathfrak{M}$-embedding $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, then we say that $\mathbf{X}$ is an *$\mathfrak{M}$-subobject* of $\mathbf{Y}$.

Given an orthogonal category $\mathfrak{K}$, a *manual in* $\mathfrak{K}$, or a *$\mathfrak{K}$-manual* for short, is a small subcategory of $\mathfrak{K}$ abiding by the following conditions:

(12) For any pair $(\mathbf{X}, \mathbf{Y})$ of objects in $\mathfrak{M}$, there exists at most a sole morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$.

(13) There exists at least a trivial object of $\mathfrak{K}$ in $\mathfrak{M}$.

(14) Every trivial object of $\mathfrak{K}$ in $\mathfrak{M}$ is $\mathfrak{M}$-trivial.

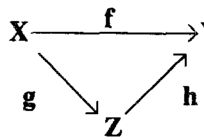
(15) For any objects $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$, if there exists a morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$, then $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$ implies $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ for any object $\mathbf{Z}$ in $\mathfrak{M}$.

(16) For any objects $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$ with $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, there exists an object $\mathbf{Z}$ of the form $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{K}$.

(17) For any object $\mathbf{Z}$ of the form $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{W}$ and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{W}$ imply $\mathbf{Z} \perp_{\mathfrak{M}} \mathbf{W}$ for any object $\mathbf{W}$ in $\mathfrak{M}$.

(18) For any objects $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$ iff there exists an object $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$, $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, and both of $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z}$ are $\mathfrak{M}$-maximal.

(19) For any commutative diagram



of $\mathfrak{K}$, if $\mathbf{f}$ is in $\mathfrak{M}$ and $\mathbf{h}$ is an $\mathfrak{M}$-embedding, then $\mathbf{g}$ is in $\mathfrak{M}$.

A $\mathfrak{K}$-manual $\mathfrak{M}$ is said to be *rich* if it satisfies the following condition:

(20) For any object $\mathbf{X}$ in $\mathfrak{M}$ and any embedding $\mathbf{f}\colon \mathbf{Y} \to \mathbf{X}$ in $\mathfrak{K}$, there exists an $\mathfrak{M}$-embedding $\mathbf{f}'\colon \mathbf{Y}' \to \mathbf{X}$ in $\mathfrak{M}$ such that $\mathbf{f}$ and $\mathbf{f}'$ are equivalent in $\mathfrak{K}$.

A $\mathfrak{K}$-manual $\mathfrak{M}$ is said to be *completely coherent* if it satisfies the following condition:

(21) For any infinite family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, there exists an object $\mathbf{Z}$ in $\mathfrak{M}$ with $\mathbf{Z} = \Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$.

Boolean Locales

The category of complete Boolean algebras and complete Boolean homomorphisms is denoted by $\mathfrak{Bool}$. The dual category of $\mathfrak{Bool}$ is denoted by $\mathfrak{BLoc}$. Its objects are called *Boolean locales.* If we regard a Boolean locale $\mathbf{X}$ as an object in $\mathfrak{Bool}$, it is often denoted by $\mathcal{P}(\mathbf{X})$ for emphasis, though $\mathbf{X}$ and $\mathcal{P}(\mathbf{X})$ denote the same entity. The opposite $\mathbf{f}^{\mathrm{op}}$ of a morphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{BLoc}$, which is a complete Boolean homomorphism from $\mathcal{P}(\mathbf{Y})$ to $\mathcal{P}(\mathbf{X})$, is usually denoted by $\mathcal{P}(\mathbf{f})$. The category $\mathfrak{BLoc}$ is cocomplete, and the pair ($\mathfrak{BLoc}$, $\mathfrak{cp}_{\mathfrak{BLoc}}$) is an orthogonal category, where $\mathfrak{cp}_{\mathfrak{BLoc}}$ is the class of coproduct diagrams in $\mathfrak{BLoc}$. Unless stated to the contrary, the category $\mathfrak{BLoc}$ is to be regarded as an orthogonal category in this sense.

A completely coherent rich manual in the orthogonal category $\mathfrak{BLoc}$ is called a *manual of Boolean locales.* A pristine example of a manual of Boolean locales can be provided by an arbitrary complete Boolean algebra $\mathbf{B}$. For each $p \in \mathbf{B}$ we denote by $\mathbf{X}_p$ the Boolean locale with $\mathcal{P}(\mathbf{X}_p) = \mathbf{B}|p = \{q \in \mathbf{B} \mid q \leq p\}$. The first-class Boolean manual $\mathfrak{M}_{\mathbf{B}}$ of Boolean locales over $\mathbf{B}$ is a subcategory of the category $\mathfrak{BLoc}$ whose objects are all $\mathbf{X}_p$ ($p \in \mathbf{B}$). A morphism $\mathbf{f}\colon \mathbf{X}_p \to \mathbf{X}_q$ of Boolean locales with $p, q \in \mathbf{B}$ lies in $\mathfrak{M}_{\mathbf{B}}$ iff $p \leq q$ and $\mathcal{P}(f)(x) = x \wedge p$ for any $x \in \mathcal{P}(\mathbf{X}_q)$.

Given a Boolean locale $\mathbf{X}$, we denote by $\Xi_{\mathbf{X}}$ the Stonean space of the complete Boolean algebra $\mathcal{P}(\mathbf{X})$. Under the so-called Stone duality, each $p \in \mathcal{P}(\mathbf{X})$ corresponds to a clopen (closed and open) subset of $\Xi_{\mathbf{X}}$, which is denoted by $\Xi_{\mathbf{X},p}$.

1. BOOLEAN SET THEORY

This section is essentially a review, and the reader is referred to Bell (1988), Goldblatt (1979), Johnstone (1977), or MacLane and Moerdijk (1992) for the general theory of topoi. For pedagogical reasons the exposition is a bit more leisurely than it should be in a technical paper. To make the paper accessible to a larger audience, we do not pursue full sophistication or full generality.

As remarked in Nishimura (1993*b*, p. 1297), every poset and every complete Boolean algebra **B** in particular can be regarded as a category. The objects of the category **B** are the elements of **B**. Given a pair (p, q) of objects of the category **B**, it is always the case that there exists at most one arrow from p to q, and there is one iff $p \leq q$. We denote by $\mathfrak{Ens}$ the category of sets and functions. A *presheaf* over **B** is a contravariant functor $\mathcal{A}$ from the category **B** to the category $\mathfrak{Ens}$, in which, given $p, q \in \mathbf{B}$ and $x \in \mathcal{A}(q)$ with $p \leq q$ so that there exists a unique arrow $f_{q,p}$ from p to q in the category **B**, we often write $\mathcal{A}_{p,q}(x)$ for $\mathcal{A}(f_{q,p})(x)$.

A (possibly empty) family $\{p_\lambda\}_{\lambda \in \Lambda}$ of nonzero elements of **B** is called a *partial partition of unity* of **B** if $p_\lambda \wedge p_{\lambda'} = 0$ for any $\lambda \neq \lambda'$. Given two partial partitions $\{p_\lambda\}_{\lambda \in \Lambda}$ and $\{q_\lambda\}_{\lambda \in \Gamma}$ of unity of **B**, the former is said to be a *refinement* of the latter if

$$\vee_{\lambda \in \Lambda} p_\lambda = \vee_{\gamma \in \Gamma} q_\lambda \tag{1.1}$$

and

$$\text{for any } \lambda \in \Lambda \text{ there exists } \gamma \in \Gamma \text{ such that } p_\lambda \leq q_\lambda \tag{1.2}$$

A presheaf $\mathcal{A}$ over **B** is called a *sheaf* over **B** if for any partial partition $\{p_\lambda\}_{\lambda \in \Lambda}$ of unity of **B** and any family $\{x_\lambda\}_{\lambda \in \Lambda}$ with $x_\lambda \in \mathcal{A}(p_\lambda)$ for each $\lambda \in \Lambda$, there exists a unique $x \in \mathcal{A}(\vee_{\lambda \in \Lambda} p_\lambda)$ with $\mathcal{A}_{p_\lambda, \vee_{\lambda \in \Lambda} p_\lambda}(x) = x_\lambda$ for each $\lambda \in \Lambda$. Every presheaf $\mathcal{A}$ over **B** has its associated sheaf $\tilde{\mathcal{A}}$ over **B**, which is called the *sheafification* of $\mathcal{A}$. For each $p \in \mathbf{B}$ we denote by $\underline{\tilde{\mathcal{A}}}(p)$ the set of all families $\{(x_\lambda, p_\lambda)\}_{\lambda \in \Lambda}$ such that

$\{p_\lambda\}_{\lambda \in \Lambda}$ is a partial partition of unity of **B** with

$$\vee_{\lambda \in \Lambda} p_\lambda = p \tag{1.3}$$

$$x_\lambda \in \mathcal{A}(p_\lambda) \text{ for each } \lambda \in \Lambda \tag{1.4}$$

Let $\tilde{\mathcal{A}}(p)$ be the quotient set of $\underline{\tilde{\mathcal{A}}}(p)$ with respect to an equivalence relation $\equiv_{\mathcal{A},p}$ on $\underline{\tilde{\mathcal{A}}}(p)$, where

$\{(x_\lambda, p_\lambda)\}_{\lambda \in \Lambda} \equiv_{\mathcal{A},p} \{(y_\lambda, q_\lambda)\}_{\gamma \in \Gamma}$ iff $\{p_\lambda\}_{\lambda \in \Lambda}$ and $\{q_\lambda\}_{\gamma \in \Gamma}$ have a common refinement $\{r_\xi\}_{\xi \in \Xi}$ such that $\mathcal{A}_{r_\xi, p_\lambda}(x_\lambda) = \mathcal{A}_{r_\xi, q_\lambda}(y_\lambda)$ whenever $r_\xi \leq p_\lambda$ and $r_\xi \leq q_\lambda$. (1.5)

We denote by $[\{(x_\lambda, p_\lambda)\}_{\lambda\in\Lambda}]_{\equiv_{\mathscr{A},p}}$ the equivalence class of an element $\{(x_\lambda, p_\lambda)\}_{\lambda\in\Lambda}$ of $\underline{\tilde{\mathscr{A}}}(p)$ with respect to $\equiv_{\mathscr{A},p}$. For any $p, q \in \mathbf{B}$ with $p \le q$, we let

$$\tilde{\mathscr{A}}_{p,q}([\{(x_\lambda, q_\lambda)\}_{\lambda\in\Lambda}]_{\equiv_{\mathscr{A},q}}) = [\{(\mathscr{A}_{q_\lambda \wedge p}(x_\lambda), q_\lambda \wedge p)\}_{\lambda\in\Lambda}]_{\equiv_{\mathscr{A},p}}$$

We denote by $\underline{\mathfrak{BSh}}(\mathbf{B})$ the category whose objects are all presheaves over $\mathbf{B}$ and whose morphisms are all natural transformations between such contravariant functors. We denote by $\mathfrak{BSh}(\mathbf{B})$ the full subcategory of $\underline{\mathfrak{BSh}}(\mathbf{B})$ whose objects are all sheaves over $\mathbf{B}$. If $\mathbf{B}$ is regarded as a Boolean locale $\mathbf{X}$, the categories $\underline{\mathfrak{BSh}}(\mathbf{B})$ and $\mathfrak{BSh}(\mathbf{B})$ are denoted also by $\underline{\mathfrak{BSh}}(\mathbf{X})$ and $\mathfrak{BSh}(\mathbf{X})$, respectively. It is easy to see that the assignment to each presheaf $\mathscr{A}$ over $\mathbf{B}$ of $\tilde{\mathscr{A}}$ and to each morphism $\mathscr{J}: \mathscr{A} \to \mathscr{B}$ in $\underline{\mathfrak{BSh}}(\mathbf{B})$ of a morphism $\tilde{\mathscr{J}}: \tilde{\mathscr{A}} \to \tilde{\mathscr{B}}$ in $\mathfrak{BSh}(\mathbf{B})$ with

$$\tilde{\mathscr{J}}_p([\{(x_\lambda, p_\lambda)\}_{\lambda\in\Lambda}]_{\equiv_{\mathscr{A},p}}) = [\{(\mathscr{J}_{p_\lambda}(x_\lambda), p_\lambda)\}_{\lambda\in\Lambda}]_{\equiv_{\mathscr{B},p}}$$

for each $p \in \mathbf{B}$ is a functor from $\underline{\mathfrak{BSh}}(\mathbf{B})$ to $\mathfrak{BSh}(\mathbf{B})$, which is denoted by $\mathrm{C}_{\mathfrak{BSh}}$. It is easy to see the following result.

Theorem 1.1. The functor $\mathrm{C}_{\mathfrak{BSh}}$: $\underline{\mathfrak{BSh}}(\mathbf{B}) \to \mathfrak{BSh}(\mathbf{B})$ is left adjoint to the inclusion functor $\mathrm{i}_{\mathfrak{BSh}}$: $\mathfrak{BSh}(\mathbf{B}) \to \underline{\mathfrak{BSh}}(\mathbf{B})$.

We now give an alternative description of the category $\mathfrak{BSh}(\mathbf{B})$. A **B**-*valued set* is a pair $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$ of a set X and a function $[\![\cdot = \cdot]\!]_X^{\mathbf{B}}: X \times X \to \mathbf{B}$ satisfying

$$[\![x = x']\!]_X^{\mathbf{B}} = [\![x' = x]\!]_X^{\mathbf{B}} \tag{1.6}$$

$$[\![x = x']\!]_X^{\mathbf{B}} \wedge [\![x' = x'']\!]_X^{\mathbf{B}} \le [\![x = x'']\!]_X^{\mathbf{B}} \tag{1.7}$$

for all $x, x', x'' \in X$. The **B**-valued set $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$ is often denoted by X unless serious confusion may arise.

Given a **B**-valued set $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$, a function $\alpha: X \to \mathbf{B}$ is called a *singleton* if it satisfies

$$\alpha(x) \wedge [\![x = x']\!]_X^{\mathbf{B}} \le \alpha(x') \tag{1.8}$$

$$\alpha(x) \wedge \alpha(x') \le [\![x = x']\!]_X^{\mathbf{B}} \tag{1.9}$$

for all $x, x' \in X$. It is easy to see that any $x \in X$ gives rise to a singleton $\{x\}$ assigning to each $x' \in X$ the set $[\![x = x']\!]_X^{\mathbf{B}} \in \mathbf{B}$. The **B**-valued set $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$ is said to be *complete* if every singleton is of the form $\{x\}$ for a unique $x \in X$. A **B**-valued set $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$, even if it is not complete, can give rise canonically to a complete **B**-valued set $(\tilde{X}, [\![\cdot = \cdot]\!]_{\tilde{X}}^{\mathbf{B}})$, where $\tilde{X}$ is the set of singletons of the **B**-valued set $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$ and $[\![\alpha = \beta]\!]_{\tilde{X}}^{\mathbf{B}} = \vee_{x\in X} (\alpha(x) \wedge \beta(x))$ for all α, β in $\tilde{X}$. The **B**-valued set $(\tilde{X}, [\![\cdot = \cdot]\!]_{\tilde{X}}^{\mathbf{B}})$ is called the *completion* of $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$. We denote by $\underline{\mathfrak{BEns}}(\mathbf{B})$ the category whose objects are all $\mathbf{B}$-valued sets and whose morphisms from a $\mathbf{B}$-valued set $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$ to another $\mathbf{B}$-valued set $(Y, [\![\cdot = \cdot]\!]_Y^{\mathbf{B}})$ are all functions $f: X \times Y \to \mathbf{B}$ satisfying

$$[\![x = x']\!]_X^{\mathbf{B}} \wedge f(x, y) \leq f(x', y) \tag{1.10}$$

$$f(x, y) \wedge [\![y = y']\!]_Y^{\mathbf{B}} \leq f(x, y') \tag{1.11}$$

$$f(x, y) \wedge f(x, y') \leq [\![y = y']\!]_Y^{\mathbf{B}} \tag{1.12}$$

$$\vee_{y \in Y} f(x, y) = [\![x = x]\!]_X^{\mathbf{B}} \tag{1.13}$$

for all $x, x' \in X$, $y, y' \in Y$. The full subcategory of $\underline{\mathfrak{BEns}}(\mathbf{B})$ whose objects are all complete $\mathbf{B}$-valued sets is denoted by $\mathfrak{BEns}(\mathbf{B})$. If the complete Boolean algebra $\mathbf{B}$ is regarded as a Boolean locale $\mathbf{X}$, then categories $\underline{\mathfrak{BEns}}(\mathbf{B})$ and $\mathfrak{BEns}(\mathbf{B})$ are denoted also by $\underline{\mathfrak{BEns}}(\mathbf{X})$ and $\mathfrak{BEns}(\mathbf{X})$, respectively. It is easy to see that the assignment to each $\mathbf{B}$-valued set $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$ of its completion $(\tilde{X}, [\![\cdot = \cdot]\!]_{\tilde{X}}^{\mathbf{B}})$ and to each morphism

$$f: \quad (X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}}) \to (Y, [\![\cdot = \cdot]\!]_Y^{\mathbf{B}}$$

of the morphism

$$\tilde{f}: \quad (\tilde{X}, [\![\cdot = \cdot]\!]_{\tilde{X}}^{\mathbf{B}}) \to (\tilde{Y}, [\![\cdot = \cdot]\!]_{\tilde{Y}}^{\mathbf{B}}$$

with

$$\tilde{f}(\alpha, \beta) = \bigvee_{\substack{x \in X \\ y \in Y}} (\alpha(x) \wedge \beta(y) \wedge f(x, y)) \qquad \text{for all } \alpha \in \tilde{X} \text{ and all } \beta \in \tilde{Y}$$

is a functor from $\underline{\mathfrak{BEns}}(\mathbf{B})$ to $\mathfrak{BEns}(\mathbf{B})$, which is denoted by $\mathfrak{C}_{\mathfrak{BEns}}$. The following result is well known.

Theorem 1.2. The functor $\mathfrak{C}_{\mathfrak{BEns}}: \underline{\mathfrak{BEns}}(\mathbf{B}) \to \mathfrak{BEns}(\mathbf{B})$ is left adjoint to the inclusion functor $\mathfrak{i}_{\mathfrak{BEns}}: \mathfrak{BEns}(\mathbf{B}) \to \underline{\mathfrak{BEns}}(\mathbf{B})$.

Now we are going to show that categories $\mathfrak{BSh}(\mathbf{B})$ and $\mathfrak{BEns}(\mathbf{B})$ are equivalent. First we define a functor $\Phi: \mathfrak{BSh}(\mathbf{B}) \to \mathfrak{BEns}(\mathbf{B})$. Let $\mathcal{A}$ be a sheaf over $\mathbf{B}$. We define a set $X_{\mathcal{A}}$ to be the disjoint union of $\mathcal{A}(p)$'s for all $p \in \mathbf{B}$. We define a function $[\![\cdot = \cdot]\!]_{X_{\mathcal{A}}}^{\mathbf{B}}: X_{\mathcal{A}} \times X_{\mathcal{A}} \to \mathbf{B}$ as follows:

$$[\![x = y]\!]_{X_{\mathcal{A}}}^{\mathbf{B}} = \sup\{r \in \mathbf{B} \mid r \leq p \wedge q \text{ and } \mathcal{A}_{r,p}(x) = \mathcal{A}_{r,q}(y)\} \tag{1.14}$$

for $x \in \mathcal{A}(p)$ and $y \in \mathcal{A}(q)$. It is easy to see that $(X_{\mathcal{A}}, [\![\cdot = \cdot]\!]_{X_{\mathcal{A}}}^{\mathbf{B}})$ is a complete $\mathbf{B}$-valued set, which shall be $\Phi(\mathcal{A})$. Let $\mathfrak{z}: \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathfrak{BSh}(\mathbf{B})$. We define a function $f_{\mathfrak{z}}: X_{\mathcal{A}} \times X_{\mathcal{B}} \to \mathbf{B}$ as follows:

$$f_{\mathfrak{z}}(x, y) = \sup\{r \in \mathbf{B} \mid r \leq p \wedge q \text{ and } \mathfrak{z}_r(\mathcal{A}_{r,p}(x)) = \mathcal{B}_{r,q}(y)\} \tag{1.15}$$

for $x \in \mathcal{A}(p)$ and $y \in \mathcal{B}(q)$. It is easy to see that $f_{\mathfrak{z}}$ is a morphism from $\Phi(\mathcal{A})$ to $\Phi(\mathcal{B})$ in $\mathfrak{BEns}(\mathbf{B})$, which shall be $\Phi(\mathfrak{z})$.

Now we will define a functor Ψ: $\mathfrak{BEns}(\mathbf{B}) \to \mathfrak{BSh}(\mathbf{B})$. Let $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$ be a complete $\mathbf{B}$-valued set. We define a presheaf $\mathscr{A}_X$ over $\mathbf{B}$ as follows:

$$\mathscr{A}_X(p) = \{x \in X \mid [\![x = x]\!]_X^{\mathbf{B}} = p\} \qquad \text{for} \quad p \in \mathbf{B} \tag{1.16}$$

For $p, q \in \mathbf{B}$ with $p \leq q$ and $x \in \mathscr{A}(q)$, $(\mathscr{A}_X)_{pq}(x)$ shall be the element of X corresponding to the singleton

$$y \in X \mapsto [\![x = y]\!]_X^{\mathbf{B}} \wedge p \tag{1.17}$$

It is easy to see that $\mathscr{A}_X$ is a sheaf over $\mathbf{B}$, which shall be $\Psi((X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}})$. Let f: $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}}) \to (Y, [\![\cdot = \cdot]\!]_Y^{\mathbf{B}})$ be a morphism in $\mathfrak{BEns}(\mathbf{B})$. For each $p \in \mathbf{B}$ we define a function $(\mathscr{s}_f)_p$: $\mathscr{A}_X(p) \to \mathscr{A}_Y(p)$ assigning to each $x \in \mathscr{A}_X(p)$ the element $(\mathscr{s}_f)_p(x)$ of Y corresponding to the singleton $y \in Y \mapsto \sup\{f(x, y') \wedge [\![y = y']\!]_Y^{\mathbf{B}} \mid y' \in Y\}$. It is easy to see that the range of $(\mathscr{s}_f)_p$ is contained in $\mathscr{A}_Y(p)$, so that $(s_f)_p$ can be considered to be a function from $\mathscr{A}_X(p)$ to $\mathscr{A}_Y(p)$. It is also easy to see that the assignment $\mathscr{s}_f$ to each $p \in \mathbf{B}$ of the function $(\mathscr{s}_f)_p$: $\mathscr{A}_X(p) \to \mathscr{A}_Y(p)$ is a morphism from $\Psi((X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}}))$ to $\Psi((Y, [\![\cdot = \cdot]\!]_Y^{\mathbf{B}}))$ in $\mathfrak{BSh}(\mathbf{B})$, which shall be $\Psi(f)$.

It is not difficult to see the following result.

Theorem 1.3. $\Psi \circ \Phi$ is naturally equivalent to the identity functor $\mathrm{I}_{\mathfrak{BSh}(\mathbf{B})}$ of the category $\mathfrak{BSh}(\mathbf{B})$, and $\Phi \circ \Psi$ is naturally equivalent to the identity functor $\mathrm{I}_{\mathfrak{BEns}(\mathbf{B})}$ of the category $\mathfrak{BEns}(\mathbf{B})$, so that categories $\mathfrak{BSh}(\mathbf{B})$ and $\mathfrak{BEns}(\mathbf{B})$ are equivalent.

The category $\mathfrak{BSh}(\mathbf{B})$ as well as the category $\mathfrak{BEns}(\mathbf{B})$ are known to constitute a Boolean localic topos, and Theorem 1.3 is only a special case of the well-known theorem that a Boolean localic topos is determined uniquely up to equivalence by the complete Boolean algebra of the elements of its subobject classifier.

Let $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ be a morphism in $\mathfrak{BLoc}$. We are now going to define functors $\mathbf{f}_*$: $\mathfrak{BSh}(\mathbf{X}) \to \mathfrak{BSh}(\mathbf{Y})$ and $\mathbf{f}_*$: $\mathfrak{BSh}(\mathbf{Y}) \to \mathfrak{BSh}(X)$, which are to be called the *direct image functor* of $\mathbf{f}$ and the *inverse image functor* of $\mathbf{f}$, respectively. It is easy to see that for any sheaf $\mathscr{A}$ over $\mathscr{P}(\mathbf{X})$, $\mathscr{A} \circ \mathscr{P}(\mathbf{f})$ is a sheaf over $\mathscr{P}(\mathbf{Y})$, which shall be $\mathbf{f}_*\mathscr{A}$. It is also easy to see that for any morphism $\mathscr{s}$: $\mathscr{A} \to \mathscr{B}$ in $\mathfrak{BSh}(\mathbf{X})$, $\mathscr{s}\mathscr{P}(\mathbf{f})$ is a morphism from $\mathbf{f}_*\mathscr{A}$ to $\mathbf{f}_*\mathscr{B}$ in $\mathfrak{BSh}(\mathbf{Y})$, which shall be $\mathbf{f}_*\mathscr{s}$.

To define the inverse image functor $\mathbf{f}^*$: $\mathfrak{BSh}(\mathbf{Y}) \to \mathfrak{BSh}(\mathbf{X})$, we first define a functor $\underline{\mathbf{f}}^*$: $\underline{\mathfrak{BEns}}(\mathbf{Y}) \to \underline{\mathfrak{BEns}}(\mathbf{X})$, which shall assign to each object $(Y, [\![\cdot = \cdot]\!]_Y^{\mathscr{P}(\mathbf{Y})})$ in $\mathfrak{BEns}(\mathbf{Y})$ the object $(Y, \mathscr{P}(\mathbf{f})([\![\cdot = \cdot]\!]_Y^{\mathscr{P}(\mathbf{Y})}))$ in $\underline{\mathfrak{BEns}}(\mathbf{X})$ and to each morphism f: $(X, [\![\cdot = \cdot]\!]_X^{\mathscr{P}(\mathbf{Y})}) \to (Y, [\![\cdot = \cdot]\!]_Y^{\mathscr{P}(\mathbf{Y})})$ in

$\underline{\mathfrak{BEns}}(\mathbf{Y})$ the morphism

$$\mathcal{P}(\mathbf{f}) \circ f: \quad (X, \mathcal{P}(\mathbf{f})(\llbracket \cdot = \cdot \rrbracket_X^{\mathcal{P}(\mathbf{Y})})) \to (Y, \mathcal{P}(\mathbf{f})(\llbracket \cdot = \cdot \rrbracket_Y^{\mathcal{P}(\mathbf{Y})}))$$

in $\underline{\mathfrak{BEns}}(\mathbf{X})$. We now define $\mathbf{f}^*$ to be $\Psi \circ \mathrm{C}_{\mathfrak{BEns}} \circ \underline{\mathbf{f}}^* \circ \Phi$.

Now it is not difficult to see the following result.

Theorem 1.4. The functor $\mathbf{f}^*$: $\mathfrak{BSh}(\mathbf{Y}) \to \mathfrak{BSh}(\mathbf{X})$ is left adjoint to the functor $\mathbf{f}_*$: $\mathfrak{BSh}(\mathbf{X}) \to \mathfrak{BSh}(\mathbf{Y})$. It is also left exact.

By using the nomenclature of topos theory, the above theorem claims that the pair $(\mathbf{f}_*, \mathbf{f}^*)$ is a geometric morphism from the topos $\mathfrak{BSh}(\mathbf{X})$ to the topos $\mathfrak{BSh}(\mathbf{Y})$.

Let $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ and $\mathbf{g}$: $\mathbf{Y} \to \mathbf{Z}$ be morphisms in $\mathfrak{BLoc}$. It is easy to see that $(\mathbf{g} \circ \mathbf{f})_* = \mathbf{g}_* \circ \mathbf{f}_*$. It is also easy to see that the functors $(\mathbf{g} \circ \mathbf{f})^*$ and $\mathbf{f}^* \circ \mathbf{g}^*$ are naturally isomorphic.

Given a Boolean locale $\mathbf{X}$, since the category $\mathfrak{BSh}(\mathbf{X})$ is a Boolean localic topos, it enjoys all classical mathematics (= mathematics based on classical logic). Now we will determine concretely sheaves $\mathcal{T}_{\mathbf{X}}$ and $\mathcal{R}_{\mathbf{X}}$ standing for the subobject classifier and the set of real numbers within $\mathfrak{BSh}(\mathbf{X})$, respectively. The sheaf $\mathcal{T}_{\mathbf{X}}$ goes as follows:

$$\mathcal{T}_{\mathbf{X}}(p) = \{(r, p) \mid r \in \mathcal{P}(\mathbf{X}) \text{ and } r \leq p\} \qquad \text{for each} \quad p \in \mathcal{P}(\mathbf{X}) \tag{1.18}$$

$$(\mathcal{T}_{\mathbf{X}})_{qp}((r, p)) = (r \wedge q, q) \qquad \text{for} \quad p, q \in \mathcal{P}(\mathbf{X}) \text{ with } q \leq p \tag{1.19}$$

The sheaf $\mathcal{R}_{\mathbf{X}}$ goes as follows:

For each $p \in \mathcal{P}(\mathbf{X})$, $\mathcal{R}_{\mathbf{X}}(p)$ is the totality of real-valued Borel functions on $\Xi_{\mathbf{X},p}$, where two real-valued Borel functions on $\Xi_{\mathbf{X},p}$ are identified so long as they coincide except some meager Borel subset of $\Xi_{\mathbf{X},p}$ (1.20)

For $p, q \in \mathcal{P}(\mathbf{X})$ with $q \leq p$, $(\mathcal{R}_{\mathbf{X}})_{qp}$ assigns to each $f \in \mathcal{R}_{\mathbf{X}}(p)$ the restriction $f|\Xi_{\mathbf{X},q}$ of f to $\Xi_{\mathbf{X},q}$ (1.21)

Let $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ be a morphism in $\mathfrak{BLoc}$. We are going to discuss the relationship between $\mathcal{T}_{\mathbf{X}}$ and $\mathcal{T}_{\mathbf{Y}}$ and that between $\mathcal{R}_{\mathbf{X}}$ and $\mathcal{R}_{\mathbf{Y}}$.

Proposition 1.5. There is a natural morphism $\mathcal{T}_{\mathbf{f}}^{\#}$: $\mathcal{T}_{\mathbf{Y}} \to \mathbf{f}_*\mathcal{T}_{\mathbf{X}}$ in $\mathfrak{BSh}(\mathbf{Y})$.

Proof. Note that $(\mathbf{f}_*\mathcal{T}_{\mathbf{X}})(p) = \mathcal{T}_{\mathbf{X}}(\mathcal{P}(\mathbf{f})(p))$ for each $p \in \mathcal{P}(\mathbf{Y})$. We set $(\mathcal{T}_{\mathbf{f}}^{\#})_p((r, p)) = (\mathcal{P}(\mathbf{f})(r), \mathcal{P}(\mathbf{f})(p))$ for each $(r, p) \in \mathcal{T}_{\mathbf{Y}}(p)$. It is easy to see that $\mathcal{T}_{\mathbf{f}}^{\#}$ is indeed a morphism from $\mathcal{T}_{\mathbf{Y}}$ to $\mathbf{f}_*\mathcal{T}_{\mathbf{X}}$ in $\mathfrak{BSh}(\mathbf{Y})$. ■

Proposition 1.6. The left adjunct $\mathcal{T}_{\#}^{\mathbf{f}}$: $\mathbf{f}^*\mathcal{T}_{\mathbf{Y}} \to \mathcal{T}_{\mathbf{X}}$ of $\mathcal{T}_{\mathbf{f}}^{\#}$ is an isomorphism in $\mathfrak{BSh}(\mathbf{X})$.

Outline of the Proof. $\Phi(\mathbf{f}^*\mathcal{T}_\mathbf{Y})$ is naturally identified with $\mathrm{C}_{\mathfrak{B}\mathfrak{Ens}}$ $(\underline{\mathbf{f}}^*(\Phi(\mathcal{T}_\mathbf{Y})))$, which is in turn identified with $\Psi(\mathcal{T}_\mathbf{X})$. To see this, note that each $(r, p) \in \Phi(\mathcal{T}_\mathbf{Y})$ determines a singleton $\alpha_{(r,p)}$ on $\underline{\mathbf{f}}^*(\Phi(\mathcal{T}_\mathbf{Y}))$ by

$$\alpha_{(r,p)}((s, q)) = (r \ominus \mathcal{P}(\mathbf{f})(s)) \wedge p \wedge \mathcal{P}(\mathbf{f})(q) \text{ for each}$$

$(s, q) \in \Phi(\mathcal{T}_\mathbf{Y})$, where $\ominus$ stands for the symmetric difference in the complete Boolean algebra $\mathcal{P}(\mathbf{X})$, and note that the underlying set of $\Phi(\mathcal{T}_\mathbf{Y})$ and that of $\underline{\mathbf{f}}^*(\Phi(\mathcal{T}_\mathbf{Y}))$ are the same (1.22)

We should note also that

$$[\![(r, p) = (r', p')]\!]^{\mathcal{P}(\mathbf{X})}_{\Phi(\mathcal{T}_\mathbf{X})} = \bigvee_{(s,q)\in\Phi(\mathcal{T}_\mathbf{Y})} \alpha_{(r,p)}((s, q)) \wedge \alpha_{(r',p')}((s, q))$$

$$\text{for all} \quad (r, p), (r', p') \in \Phi(\mathcal{T}_\mathbf{X}) \tag{1.23}$$

It is not difficult to see that $\Phi(\mathcal{T}^\mathbf{f}_\#)$ renders this identification between $\mathrm{C}_{\mathfrak{B}\mathfrak{Ens}}$ $(\underline{\mathbf{f}}^*(\Phi(\mathcal{T}_\mathbf{Y})))$ and $\Psi(\mathcal{T}_\mathbf{X})$. Therefore $\mathcal{T}^\mathbf{f}_\#$ is an isomorphism.

Proposition 1.7. There is a natural morphism $\mathcal{R}^\#_\mathbf{f}$: $\mathcal{R}_\mathbf{Y} \to \mathbf{f}_*\mathcal{R}_\mathbf{X}$ in $\mathfrak{B}\mathfrak{Sh}(\mathbf{Y})$.

Proof. Note that $(\mathbf{f}_*\mathcal{R}_\mathbf{X})(p) = \mathcal{R}_\mathbf{X}(\mathcal{P}(\mathbf{f})(p))$ for each $p \in \mathcal{P}(\mathbf{Y})$. We set $(\mathcal{R}^\#_\mathbf{f})_p(f) = f \circ \xi_{\mathbf{X},\mathbf{Y},p}$ for each $f \in \mathcal{R}_\mathbf{Y}(p)$. It is easy to see that $\mathcal{R}^\#_\mathbf{f}$ is indeed a morphism from $\mathcal{R}_\mathbf{Y}$ to $\mathbf{f}_*\mathcal{R}_\mathbf{X}$ in $\mathfrak{B}\mathfrak{Sh}(\mathbf{Y})$. ■

In the next section we will use such self-explanatory notations as $\mathrm{Hom}_\mathbf{X}(\mathcal{A}, \mathcal{B})$ for the totality of morphisms from $\mathcal{A}$ to $\mathcal{B}$ in $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$.

2. EMPIRICAL SET THEORY

To begin with, we define a category to be denoted by $\mathfrak{B}\mathfrak{Ens}$. Its objects are all pairs $(\mathbf{X}, \mathcal{A})$ of a Boolean locale $\mathbf{X}$ and a sheaf $\mathcal{A}$ over the complete Boolean algebra $\mathcal{P}(\mathbf{X})$. Given two such pairs $(\mathbf{X}, \mathcal{A})$ and $(\mathbf{Y}, \mathcal{B})$, the morphisms from $(\mathbf{X}, \mathcal{A})$ to $(\mathbf{Y}, \mathcal{B})$ in $\mathfrak{B}\mathfrak{Ens}$ are all pairs $(\mathbf{f}, \mathbf{f}^\#)$ of a morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{B}\mathfrak{Loc}$ and a morphism $\mathbf{f}^\#$: $\mathcal{B} \to \mathbf{f}_*\mathcal{A}$ in $\mathfrak{B}\mathfrak{Sh}(\mathbf{Y})$. By dint of the canonical adjunction $\mathrm{Hom}_\mathbf{X}(\mathbf{f}^*\mathcal{B}, \mathcal{A}) \cong \mathrm{Hom}_\mathbf{Y}(\mathcal{B}, \mathbf{f}_*\mathcal{A})$, the morphism $(\mathbf{f}, \mathbf{f}^\#)$: $(\mathbf{X}, \mathcal{A}) \to (\mathbf{Y}, \mathcal{B})$ can be represented also by $(\mathbf{f}, \mathbf{f}_\#)$, where $\mathbf{f}_\#$: $\mathbf{f}^*\mathcal{B} \to \mathcal{A}$ is the morphism in $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$ corresponding to the morphism $\mathbf{f}^\#$: $\mathcal{B} \to \mathbf{f}^*\mathcal{A}$ in the above adjunction. The corresponding representations $(\mathbf{f}, \mathbf{f}^\#)$ and $(\mathbf{f}, \mathbf{f}_\#)$ of the same morphism are called the *upper* and *lower representations*, and they will be used complementarily according to the context. Given morphisms $(\mathbf{f}, \mathbf{f}^{\#})$: $(\mathbf{X}, \mathcal{A}) \to (\mathbf{Y}, \mathcal{B})$ and $(\mathbf{g}, \mathbf{g}^{\#})$: $(\mathbf{Y}, \mathcal{B}) \to (\mathbf{Z}, \mathcal{C})$ in $\mathfrak{BEns}$, their composition $(\mathbf{g}, \mathbf{g}^{\#}) \circ (\mathbf{f}, \mathbf{f}^{\#})$ is defined to be $(\mathbf{g} \circ \mathbf{f}, (\mathbf{g}_*\mathbf{f}^{\#}) \circ \mathbf{g}^{\#})$, where we loosely identify $\mathbf{g}_*(\mathbf{f}^*\mathcal{A})$ with $(\mathbf{g} \circ \mathbf{f})_*\mathcal{A}$. As for the lower representation of composition of morphisms in $\mathfrak{BEns}$, we have the following result.

Proposition 2.1. If $(\mathbf{f}, \mathbf{f}^{\#})$: $(\mathbf{X}, \mathcal{A}) \to (\mathbf{Y}, \mathcal{B})$ and $(\mathbf{g}, \mathbf{g}^{\#})$: $(\mathbf{Y}, \mathcal{B}) \to (\mathbf{Z}, \mathcal{C})$ are represented lowerly by $(\mathbf{f}, \mathbf{f}_{\#})$ and $(\mathbf{g}, \mathbf{g}_{\#})$, respectively, then their composition $(\mathbf{g}, \mathbf{g}_{\#}) \circ (\mathbf{f}, \mathbf{f}_{\#})$ is represented lowerly by $(\mathbf{g} \circ \mathbf{f}, \mathbf{f}_{\#} \circ (\mathbf{f}^*\mathbf{g}_{\#}))$, where we loosely identify $\mathbf{f}^*(\mathbf{g}^*\mathcal{C})$ and $(\mathbf{g} \circ \mathbf{f})^*\mathcal{C}$.

Proof. By chasing $\mathbf{f}_{\#}$ around the commutative square

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{X}}(\mathbf{f}^*\mathbf{g}^*\mathcal{C}, \mathcal{A}) & \cong & \mathrm{Hom}_{\mathbf{Y}}(\mathbf{g}^*\mathcal{C}, \mathbf{f}_*\mathcal{A}) \\
\mathrm{Hom}_{\mathbf{X}}(\mathbf{f}^*\mathbf{g}_{\#}, \mathcal{A}) \uparrow & & \uparrow \mathrm{Hom}_{\mathbf{Y}}(\mathbf{g}_{\#}, \mathbf{f}_*\mathcal{A}) \\
\mathrm{Hom}_{\mathbf{X}}(\mathbf{f}^*\mathcal{B}, \mathcal{A}) & \cong & \mathrm{Hom}_{\mathbf{Y}}(\mathcal{B}, \mathbf{f}_*\mathcal{A})
\end{array}
$$

we have

$$
\begin{array}{ccc}
\mathbf{f}_{\#}\circ(\mathbf{f}^*\mathbf{g}_{\#}) & \longmapsto & \mathbf{f}^{\#}\circ\mathbf{g}_{\#} \\
\uparrow & & \uparrow \\
\mathbf{f}_{\#} & \longmapsto & \mathbf{f}^{\#}
\end{array}
$$

By chasing $\mathbf{g}_{\#}$ around the commutative square

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Y}}(\mathbf{g}^*\mathcal{C}, \mathcal{B}) & \cong & \mathrm{Hom}_{\mathbf{Z}}(\mathcal{C}, \mathbf{g}_*\mathcal{B}) \\
\mathrm{Hom}_{\mathbf{Y}}(\mathbf{g}^*\mathcal{C}, \mathbf{f}^{\#}) \downarrow & & \downarrow \mathrm{Hom}_{\mathbf{Z}}(\mathcal{C}, \mathbf{g}_*, \mathbf{f}^{\#}) \\
\mathrm{Hom}_{\mathbf{Y}}(\mathbf{g}^*\mathcal{C}, \mathbf{f}_*\mathcal{A}) & \cong & \mathrm{Hom}_{\mathbf{Z}}(\mathcal{C}, \mathbf{g}_*\mathbf{f}_*\mathcal{A})
\end{array}
$$

we have

$$
\begin{array}{ccc}
\mathbf{g}_{\#} & \longmapsto & \mathbf{g}^{\#} \\
\downarrow & & \downarrow \\
\mathbf{f}^{\#}\circ\mathbf{g}_{\#} & \longmapsto & (\mathbf{g}_*\mathbf{f}^{\#})\circ\mathbf{g}^{\#}
\end{array}
$$

Thus $\mathbf{f}_{\#} \circ (\mathbf{f}^*\mathbf{g}_{\#})$ corresponds to $(\mathbf{g}_*\mathbf{f}^{\#}) \circ \mathbf{g}^{\#}$ under the canonical adjunction $\mathrm{Hom}_{\mathbf{X}}(\mathbf{f}^*\mathbf{g}^*\mathcal{C}, \mathcal{A}) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathcal{C}, \mathbf{g}_*\mathbf{f}_*\mathcal{A})$. ■

We denote by Θ the forgetful functor from $\mathfrak{BEns}$ to $\mathfrak{BLoc}$. That is to say, $\Theta(\mathbf{X}, \mathcal{A}) = \mathbf{X}$ for any object $(\mathbf{X}, \mathcal{A})$ in $\mathfrak{BEns}$ and $\Theta(\mathbf{f}, \mathbf{f}^{\#}) = \mathbf{f}$ for any morphism $(\mathbf{f}, \mathbf{f}^{\#})$ in $\mathfrak{BEns}$.

As in Nishimura (1995*b*, Proposition 3.2), it is easy to see the following result.

Proposition 2.2. $(\mathfrak{BEns}, \mathfrak{cp}_{\mathfrak{BEns}})$ is an orthogonal category, where $\mathfrak{cp}_{\mathfrak{BEns}}$ denotes the class of coproduct diagrams in $\mathfrak{BEns}$.

Outline of the Proof. Here we give only the coproduct construction in the category $\mathfrak{BEns}$. Let $\{(\mathbf{X}_\lambda, \mathscr{A}_\lambda)\}_{\lambda \in \Lambda}$ be a small family of objects in $\mathfrak{BEns}$. We note that the coproduct of the family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ in $\mathfrak{BLoc}$ is given by $\mathbf{X}$ with $\mathscr{P}(\mathbf{X}) = \Pi_{\lambda \in \Lambda} \mathscr{P}(\mathbf{X}_\lambda)$. The desired coproduct of $\{(\mathbf{X}_\lambda, \mathscr{A}_\lambda)\}_{\lambda \in \Lambda}$ in $\mathfrak{BEns}$ is given by $(\mathbf{X}, \mathscr{A})$, where $\mathscr{A}((p_\lambda)_{\lambda \in \Lambda}) = \Pi_{\lambda \in \Lambda} \mathscr{A}_\lambda(p_\lambda)$ for any $(p_\lambda)_{\lambda \in \Lambda}$ in $\mathscr{P}(\mathbf{X})$. ■

In the remainder of this paper the category $\mathfrak{BEns}$ is to be regarded as an orthogonal category in the above sense unless stated to the contrary.

Given a manual $\mathfrak{M}$ of Boolean locales, we now define a category to be denoted by $\mathfrak{EEns}(\mathfrak{M})$. Its objects are all functors $\mathfrak{F}: \mathfrak{M} \to \mathfrak{BEns}$ satisfying the following conditions:

(2.1) It maps orthogonal $\mathfrak{M}$-sum diagrams to orthogonal diagrams in $\mathfrak{BEns}$.

(2.2) $\Theta \circ \mathfrak{F}$ is the identity functor.

Let $\mathfrak{F}$ be such a functor. For any Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, if $\mathfrak{F}(\mathbf{X}) = (\mathbf{X}, \mathscr{A})$, then $\mathscr{A}$ is denoted by $\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})$. For any morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, if $\mathfrak{F}(\mathbf{f}) = (\mathbf{f}, \mathbf{f}^\#)$, then $\mathbf{f}^\#: \mathfrak{F}_{\mathrm{Sh}}(\mathbf{Y}) \to \mathbf{f}^*\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})$ is denoted by $\mathfrak{F}^\#(\mathbf{f})$ and $\mathbf{f}_\#: \mathbf{f}^*\mathfrak{F}_{\mathrm{Sh}}(\mathbf{Y}) \to \mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})$ is denoted by $\mathfrak{F}_\#(\mathbf{f})$.

Given such functors $\mathfrak{F}$ and $\mathfrak{G}$, morphisms from $\mathfrak{F}$ to $\mathfrak{G}$ in $\mathfrak{EEns}(\mathfrak{M})$ are all assignments φ to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of a morphism $\varphi_{\mathbf{X}}$: $\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) \to \mathfrak{G}_{\mathrm{Sh}}(\mathbf{X})$ in $\mathfrak{BSh}(\mathbf{X})$ satisfying the following condition:

$(2.3)^\#$ For any morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, the diagram

$$\begin{array}{ccc}
\mathfrak{F}_{\mathrm{Sh}}(\mathbf{Y}) & \xrightarrow{\mathfrak{F}^\#(\mathbf{f})} & \mathbf{f}_*\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) \\
\varphi_{\mathbf{Y}} \downarrow & & \downarrow \mathbf{f}_*\varphi_{\mathbf{X}} \\
\mathfrak{G}_{\mathrm{Sh}}(\mathbf{Y}) & \xrightarrow[\mathfrak{G}^\#(\mathbf{g})]{} & \mathbf{f}_*\mathfrak{G}_{\mathrm{Sh}}(\mathbf{X})
\end{array}$$

is commutative.

Proposition 2.3. The above condition $(2.3)^\#$ is equivalent to its following lower version:

$(2.3)_\#$ For any morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, the diagram

$$\begin{array}{ccc} \mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}) & \xrightarrow{\mathfrak{F}_\#(\mathbf{f})} & \mathfrak{F}_{Sh}(\mathbf{X}) \\ {\scriptstyle \mathbf{f}^*\varphi_{\mathbf{Y}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_{\mathbf{X}}} \\ \mathbf{f}^*\mathfrak{G}_{Sh}(\mathbf{Y}) & \xrightarrow[\mathfrak{G}_\#(\mathbf{f})]{} & \mathfrak{G}_{Sh}(\mathbf{X}) \end{array}$$

is commutative.

Proof. The equivalence of $(2.3)^\#$ and $(2.3)_\#$ follows readily from the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Y}}(\mathfrak{F}_{Sh}(\mathbf{Y}), \mathbf{f}_*\mathfrak{F}_{Sh}(\mathbf{X})) & \cong & \mathrm{Hom}_{\mathbf{X}}(\mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}), \mathfrak{F}_{Sh}(\mathbf{X})) \\ {\scriptstyle \mathrm{Hom}_{\mathbf{Y}}(\mathfrak{F}_{Sh}(\mathbf{Y}), \mathbf{f}_*\varphi_{\mathbf{X}})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{X}}(\mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}), \varphi_{\mathbf{X}})} \\ \mathrm{Hom}_{\mathbf{Y}}(\mathfrak{F}_{Sh}(\mathbf{Y}), \mathbf{f}_*\mathfrak{G}_{Sh}(\mathbf{X})) & \cong & \mathrm{Hom}_{\mathbf{X}}(\mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}), \mathfrak{G}_{Sh}(\mathbf{X})) \\ {\scriptstyle \mathrm{Hom}_{\mathbf{Y}}(\varphi_{\mathbf{Y}}, \mathbf{f}_*\mathfrak{G}_{Sh}(\mathbf{X}))}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{Hom}_{\mathbf{X}}(\mathbf{f}^*\varphi_{\mathbf{Y}}, \mathfrak{G}_{Sh}(\mathbf{X}))} \\ \mathrm{Hom}_{\mathbf{Y}}(\mathfrak{G}_{Sh}(\mathbf{Y}), \mathbf{f}_*\mathfrak{G}_{Sh}(\mathbf{X})) & \cong & \mathrm{Hom}_{\mathbf{X}}(\mathbf{f}^*\mathfrak{G}_{Sh}(\mathbf{Y}), \mathfrak{G}_{Sh}(\mathbf{X})) \end{array} \quad \blacksquare$$

The composition $\psi \circ \varphi$ of morphisms φ: $\mathfrak{F} \to \mathfrak{G}$ and ψ: $\mathfrak{G} \to \mathfrak{H}$ in $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M})$ is defined to be the assignment to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of $\psi_{\mathbf{X}} \circ \varphi_{\mathbf{X}}$. For each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ we denote by $\Delta_{\mathbf{X}}$ the forgetful functor from $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M})$ to $\mathfrak{B}\mathfrak{S}\mathfrak{h}(\mathbf{X})$, which assigns to each object $\mathfrak{F}$ in $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M})$ the object $\mathfrak{F}_{Sh}(\mathbf{X})$ in $\mathfrak{B}\mathfrak{S}\mathfrak{h}(\mathbf{X})$ and to each morphism φ: $\mathfrak{F} \to \mathfrak{G}$ in $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M})$ the morphism $\varphi_{\mathbf{X}}$: $\mathfrak{F}_{Sh}(\mathbf{X}) \to \mathfrak{G}_{Sh}(\mathbf{X})$ in $\mathfrak{B}\mathfrak{S}\mathfrak{h}(\mathbf{X})$.

Each object of $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M})$ is called an *empirical set over* $\mathfrak{M}$, and the category $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M})$ is called the *empirical set theory over* $\mathfrak{M}$.

Before embarking upon the general theory of empirical set theory, we present some examples of empirical set theories and empirical sets, which will put down our midair notions onto earth.

Example 2.4. Let $\mathbf{B}$ be a complete Boolean algebra and $\mathfrak{M}_{\mathbf{B}}$ the first-class Boolean manual of Boolean locales over $\mathbf{B}$. It is easy to see that the category $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M}_{\mathbf{B}})$ is equivalent to the category $\mathfrak{B}\mathfrak{S}\mathfrak{h}(\mathbf{B})$.

This example shows that our empirical set theory $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M})$ over an arbitrary manual $\mathfrak{M}$ of Boolean locales is a natural generalization of Boolean set theory.

We will keep $\mathfrak{M}$ denoting an arbitrarily chosen but fixed manual of Boolean locales up to the very end of this section.

Example 2.5. We will define an empirical set $\mathfrak{T}$ over $\mathfrak{M}$, which is intended to stand for the empirical set of truth values within the empirical set theory $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$. For each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, we set $\mathfrak{T}(\mathbf{X}) = \mathcal{T}_{\mathbf{X}}$. For each morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, Proposition 1.5 shows that there is a natural isomorphism $\mathcal{T}_{\mathbf{f}}^{\#}: \mathcal{T}_{\mathbf{Y}} \to \mathbf{f}_*\mathcal{T}_{\mathbf{X}}$, which shall be taken as $\mathfrak{T}^{\#}(\mathbf{f})$. It is not difficult to see that $\mathfrak{T}$ is indeed a well-defined empirical set over $\mathfrak{M}$.

Example 2.6. We will define an empirical set $\mathfrak{R}$ over $\mathfrak{M}$, which is intended to represent the empirical set of real numbers within the empirical set theory $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$. For each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, we set $\mathcal{R}(\mathbf{X}) = \mathcal{R}_{\mathbf{X}}$. For each morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, Proposition 1.7 shows that there is a natural isomorphism $\mathcal{R}_{\mathbf{f}}^{\#}: \mathcal{R}_{\mathbf{Y}} \to \mathbf{f}_*\mathcal{R}_{\mathbf{X}}$, which shall be taken as $\mathcal{R}^{\#}(\mathbf{f})$. It is not difficult to see that $\mathcal{R}$ is indeed a well-defined empirical set over $\mathfrak{M}$.

Now we are going to show that, roughly speaking, the category $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$ is not a topos only in that exponentials do not exist in general.

Proposition 2.7. The category $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$ has products for any small family of objects.

Proof. Let $\{\mathfrak{F}_\lambda\}_{\lambda\in\Lambda}$ be a small family of objects in $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$. For any Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, since $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$ is complete, there exists a product diagram

$$\{\mathcal{A}_{\mathbf{X}} \xrightarrow{(\varphi_\lambda)_{\mathbf{X}}} (\mathfrak{F}_\lambda)_{\mathrm{Sh}}(\mathbf{X})\}_{\lambda\in\Lambda} \qquad \text{in } \mathfrak{B}\mathfrak{Sh}(\mathbf{X})$$

For any morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, since the functor $\mathbf{f}_*: \mathfrak{B}\mathfrak{Sh}(\mathbf{X}) \to \mathfrak{B}\mathfrak{Sh}(\mathbf{Y})$ preserves limits,

$$\{\mathbf{f}_*\mathcal{A}_{\mathbf{X}} \xrightarrow{\mathbf{f}_*(\varphi_\lambda)_{\mathbf{X}}} \mathbf{f}_*(\mathfrak{F}_\lambda)_{\mathrm{Sh}}(\mathbf{X})\}_{\lambda\in\Lambda}$$

is a product diagram in $\mathfrak{B}\mathfrak{Sh}(\mathbf{Y})$. Therefore there exists a unique morphism $\mathbf{f}^{\#}: \mathcal{A}_{\mathbf{Y}} \to \mathbf{f}_*\mathcal{A}_{\mathbf{X}}$ in $\mathfrak{B}\mathfrak{Sh}(\mathbf{Y})$ such that $(\mathfrak{F}_\lambda)^{\#}(\mathbf{f}) \circ (\varphi_\lambda)_{\mathbf{Y}} = (\mathbf{f}^*(\varphi_\lambda)_{\mathbf{X}}) \circ \mathbf{f}^{\#}$ for any $\lambda \in \Lambda$. It is not difficult to see that there exists a unique object $\mathfrak{F}$ in $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$ such that $\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) = \mathcal{A}_{\mathbf{X}}$ for any Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ and $\mathfrak{F}^{\#}(\mathbf{f}) = \mathbf{f}^{\#}$ for any morphism $\mathbf{f}$ in $\mathfrak{M}$. It is easy to see that the assignment φ_λ to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of $(\varphi_\lambda)_{\mathbf{X}}$ is a morphism from $\mathfrak{F}$ to $\mathfrak{F}_\lambda$ for each $\lambda \in \Lambda$. Now it remains to show that for any family $\{\mathfrak{G} \to^{\psi_\lambda} \mathfrak{F}_\lambda\}_{\lambda\in\Lambda}$ of morphisms in $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$, there exists a unique morphism $\chi: \mathfrak{G} \to \mathfrak{F}$ with $\varphi_\lambda \circ \chi = \psi_\lambda$ for any $\lambda \in \Lambda$. Since

$$\{\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) \xrightarrow{(\varphi_\lambda)_{\mathbf{X}}} (\mathfrak{F}_\lambda)_{\mathrm{Sh}}(\mathbf{X})\}_{\lambda\in\Lambda}$$

is a product diagram in $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$ by definition for each Boolean locale $\mathbf{X}$ in

$\mathfrak{M}$, there exists a unique morphism $\chi_X: \mathfrak{G}_{Sh}(X) \to \mathfrak{F}_{Sh}(X)$ in $\mathfrak{BSh}(X)$ with $(\varphi_\lambda)_X \circ \chi_X = (\psi_\lambda)_X$ for all $\lambda \in \Lambda$. Thus it suffices to show that for any morphism $\mathbf{f}: X \to Y$ in $\mathfrak{M}$, $\mathfrak{F}^\#(\mathbf{f}) \circ \chi_Y = (\mathbf{f}_*\chi_X) \circ \mathfrak{G}^\#(\mathbf{f})$, for the assignment χ to each Boolean locale X in $\mathfrak{M}$ of χ_X would be the desired morphism in $\mathfrak{FEns}(\mathfrak{M})$. For each $\lambda \in \Lambda$ we have a commutative diagram

$$\begin{array}{ccccc}
 & \mathfrak{F}_{Sh}(Y) & \xrightarrow{\mathfrak{F}^\#(\mathbf{f})} & \mathbf{f}_*\mathfrak{F}_{Sh}(X) & \\
 & {\scriptstyle (\varphi_\lambda)_Y}\downarrow & & \downarrow{\scriptstyle \mathbf{f}_*(\varphi_\lambda)_X} & \\
\chi_Y & (\mathfrak{F}_\lambda)_{Sh}(Y) & \xrightarrow{\mathfrak{F}_\lambda(\mathbf{f})} & \mathbf{f}_*(\mathfrak{F}_\lambda)_{Sh}(X) & \mathbf{f}_*\chi_X \\
 & {\scriptstyle (\psi_\lambda)_Y}\uparrow & & \uparrow{\scriptstyle \mathbf{f}_*(\psi_\lambda)_X} & \\
 & \mathfrak{G}_{Sh}(Y) & \xrightarrow{\mathfrak{G}^\#(\mathbf{f})} & \mathbf{f}_*\mathfrak{G}_{Sh}(X) &
\end{array}$$

in $\mathfrak{BSh}(Y)$, for which we get

$$\begin{aligned}
&(\mathbf{f}_*(\varphi_\lambda)_X) \circ (\mathbf{f}_*\chi_X) \circ \mathfrak{G}^\#(\mathbf{f}) \\
&= (\mathbf{f}^*(\psi_\lambda)_X) \circ \mathfrak{G}^\#(\mathbf{f}) \\
&= (\mathfrak{F}_\lambda)^\#(\mathbf{f}) \circ (\psi_\lambda)_Y \\
&= (\mathfrak{F}_\lambda)^\# \circ (\varphi_\lambda)_Y \circ \chi_Y \\
&= (\mathbf{f}_*(\varphi_\lambda)_X) \circ \mathfrak{F}^\#(\mathbf{f}) \circ \chi_Y
\end{aligned}$$

Since

$$\{\mathbf{f}_*\mathfrak{F}_{Sh}(X) \xrightarrow{\mathbf{f}_*(\varphi_\lambda)_X} \mathbf{f}_*(\mathfrak{F}_\lambda)_{Sh}(X)\}_{\lambda \in \Lambda}$$

is a product diagram in $\mathfrak{BSh}(Y)$, $\mathfrak{F}^\#(\mathbf{f}) \circ \chi_Y = (\mathbf{f}_*\chi_X) \circ \mathfrak{G}^\#(\mathbf{f})$, which was the desired equality. ∎

By the same token, we have the following result.

Proposition 2.8. The category $\mathfrak{FEns}(\mathfrak{M})$ has equalizers for any parallel morphisms.

Theorem 2.9. The category $\mathfrak{FEns}(\mathfrak{M})$ is complete.

Proof. It is well known that a category of products and equalizers is complete, for which the reader is referred, e.g., to MacLane (1971, Chapter V, §2). Therefore the desired result follows from Propositions 2.7 and 2.8. ∎

The proofs of Propositions 2.7 and 2.8 give also the following result.

Proposition 2.10. Given a diagram $F: J \to \mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$ and a cone $\tau: \mathfrak{F} \dot{\to} F$ in $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$, τ is a limiting cone iff the cone $\Delta_{\mathbf{X}} \circ \tau: \Delta_{\mathbf{X}}(\mathfrak{F}) \dot{\to} \Delta_{\mathbf{X}} \circ F$ is a limiting cone in $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$ for any object $\mathbf{X}$ in $\mathfrak{M}$.

The discussion from Proposition 2.7 through Proposition 2.10 can be dualized.

Proposition 2.11. The category $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$ has coproducts for any small family of objects.

Proof. Let $\{\mathfrak{F}_\lambda\}_{\lambda \in \Lambda}$ be a small family of objects in $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$. Let $\mathbf{X}$ be a Boolean locale in $\mathfrak{M}$. Since $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$ is cocomplete, there exists a coproduct diagram

$$\{(\mathfrak{F}_\lambda)_{\mathrm{Sh}}(\mathbf{X}) \xrightarrow{(\varphi_\lambda)_{\mathbf{X}}} \mathcal{A}_{\mathbf{X}}\}_{\lambda \in \Lambda}$$

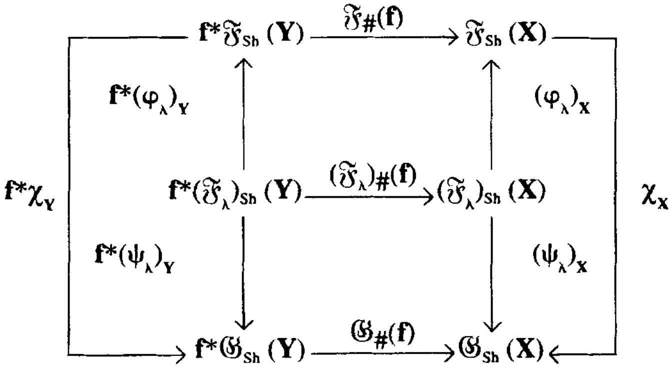
in $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$. Let $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ be a morphism in $\mathfrak{M}$. Since the functor $\mathbf{f}^*: \mathfrak{B}\mathfrak{Sh}(\mathbf{Y}) \to \mathfrak{B}\mathfrak{Sh}(\mathbf{X})$ preserves colimits,

$$\{\mathbf{f}^*(\mathfrak{F}_\lambda)_{\mathrm{Sh}}(\mathbf{Y}) \xrightarrow{\mathbf{f}^*(\varphi_\lambda)_{\mathbf{Y}}} \mathbf{f}^*\mathcal{A}_{\mathbf{Y}}\}_{\lambda \in \Lambda}$$

is a coproduct diagram in $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$. Therefore there exists a unique morphism $\mathbf{f}_\#: \mathbf{f}^*\mathcal{A}_{\mathbf{Y}} \to \mathcal{A}_{\mathbf{X}}$ in $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$ such that $(\varphi_\lambda)_{\mathbf{X}} \circ (\mathfrak{F}_\lambda)_\#(\mathbf{f}) = \mathbf{f}_\# \circ (\mathbf{f}^*(\varphi_\lambda)_{\mathbf{Y}})$ for any $\lambda \in \Lambda$. It is not difficult to see that there exists a unique object $\mathfrak{F}$ in $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$ such that $\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) = \mathcal{A}_{\mathbf{X}}$ for any Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ and $\mathfrak{F}_\#(\mathbf{f}) = \mathbf{f}_\#$ for any morphism $\mathbf{f}$ in $\mathfrak{M}$. It is easy to see that the assignment φ_λ to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of $(\varphi_\lambda)_{\mathbf{X}}$ is a morphism from $\mathfrak{F}_\lambda$ to $\mathfrak{F}$ for each $\lambda \in \Lambda$. Now it remains to show that for any family $\{\mathfrak{F}_\lambda \to^{\psi_\lambda} \mathfrak{G}\}_{\lambda \in \Lambda}$ of morphisms in $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$, there exists a unique morphism $\chi: \mathfrak{F} \to \mathfrak{G}$ with $\chi \circ \varphi_\lambda = \psi_\lambda$ for any $\lambda \in \Lambda$. Since

$$\{(\mathfrak{F}_\lambda)_{\mathrm{Sh}}(\mathbf{X}) \xrightarrow{(\varphi_\lambda)_{\mathbf{X}}} \mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})\}_{\lambda \in \Lambda}$$

is a coproduct diagram in $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$ by definition for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, there exists a unique morphism $\chi_{\mathbf{X}}: \mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) \to \mathfrak{G}_{\mathrm{Sh}}(\mathbf{X})$ in $\mathfrak{B}\mathfrak{Sh}(\mathbf{X})$ with $\chi_{\mathbf{X}} \circ (\varphi_\lambda)_{\mathbf{X}} = (\psi_\lambda)_{\mathbf{X}}$ for all $\lambda \in \Lambda$. Thus it suffices to show that for any morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, $\chi_{\mathbf{X}} \circ \mathfrak{F}_\#(\mathbf{f}) = \mathfrak{G}_\#(\mathbf{f}) \circ (\mathbf{f}^*\chi_{\mathbf{Y}})$, for the assignment χ to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of $\chi_{\mathbf{X}}$ would be the desired morphism in $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$. For each $\lambda \in \Lambda$ we have a commutative diagram



in $\mathfrak{BSh}(\mathbf{X})$, for which we get

$$\begin{aligned}
&\mathfrak{G}^\#(\mathbf{f}) \circ (\mathbf{f}^*\chi_\mathbf{Y}) \circ (\mathbf{f}^*(\varphi_\lambda)_\mathbf{X}) \\
&= \mathfrak{G}^\#(\mathbf{f}) \circ (\mathbf{f}^*(\psi_\lambda)_\mathbf{Y}) \\
&= (\psi_\lambda)_\mathbf{X} \circ (\mathfrak{F}_\lambda)_\#(\mathbf{f}) \\
&= \chi_\mathbf{X} \circ (\varphi_\lambda)_\mathbf{X} \circ (\mathfrak{F}_\lambda)_\# \\
&= \chi_\mathbf{X} \circ \mathfrak{F}_\#(\mathbf{f}) \circ (\mathbf{f}^*(\varphi_\lambda)_\mathbf{Y})
\end{aligned}$$

Since

$$\{\mathbf{f}^*(\mathfrak{F}_\lambda)_{Sh}(\mathbf{Y}) \xrightarrow{\mathbf{f}^*(\varphi_\lambda)_\mathbf{Y}} \mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y})\}_{\lambda\in\Lambda}$$

is a coproduct diagram in $\mathfrak{BSh}(\mathbf{X})$, $\chi_\mathbf{X} \circ \mathfrak{F}_\#(\mathbf{f}) \circ = \mathfrak{G}_\#(\mathbf{f}) \circ (\mathbf{f}^*\chi_\mathbf{Y})$, which was the desired equality. ∎

By the same token, we have the following result.

Proposition 2.12. The category $\mathfrak{SEns}(\mathfrak{M})$ has coequalizers for any parallel morphisms.

Just as Propositions 2.7 and 2.8 led to Theorem 2.9, Propositions 2.11 and 2.12 lead to the following result.

Theorem 2.13. The category $\mathfrak{SEns}(\mathfrak{M})$ is cocomplete.

The proofs of Propositions 2.11 and 2.12 establish the following.

Proposition 2.14. Given a diagram $F: J \to \mathfrak{SEns}(\mathfrak{M})$ and a cone $\tau: F \dot{\to} \mathfrak{F}$ in $\mathfrak{SEns}(\mathfrak{M})$, τ is a limiting cone iff the cone $\Delta_\mathbf{X} \circ \tau: \Delta_\mathbf{X} \cdot F \dot{\to} \Delta_\mathbf{X}(\mathfrak{F})$ is a limiting cone in $\mathfrak{BSh}(\mathbf{X})$ for any object $\mathbf{X}$ in $\mathfrak{M}$.

For each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, we fix a terminal object $\mathit{1}_{\mathbf{X}}$ in the category $\mathfrak{BSh}(\mathbf{X})$ and denote by $\top_{\mathbf{X}}$ the truth arrow $\mathit{1}_{\mathbf{X}} \to \mathcal{T}_{\mathbf{X}}$ in the topos $\mathfrak{BSh}(\mathbf{X})$. We denote by $\mathbf{1}$ the terminal object of $\mathfrak{EEns}(\mathfrak{M})$ with $\mathbf{1}_{\mathrm{Sh}}(\mathbf{X}) = \mathit{1}_{\mathbf{X}}$ for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$. The assignment to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of $\top_{\mathbf{X}}$ is easily seen to be a morphism $\mathbf{1} \to \mathfrak{T}$, and is denoted by $\top$. The rest of this section is consecrated to showing that the morphism $\top$: $\mathbf{1} \to \mathfrak{T}$ plays a role of a subobject classifier for a well-behaved class of subobjects, and that exponentials exist for highly degenerative empirical sets. To this end, we first need the following result.

Proposition 2.15. A morphism φ: $\mathfrak{F} \to \mathfrak{G}$ in $\mathfrak{EEns}(\mathfrak{M})$ is a monomorphism iff $\varphi_{\mathbf{X}}$: $\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) \to \mathfrak{G}_{\mathrm{Sh}}(\mathbf{X})$ is a monomorphism in $\mathfrak{BSh}(\mathbf{X})$ for every Boolean locale $\mathbf{X}$ in $\mathfrak{M}$.

Proof. We know well (cf. Schubert, 1972, 7.8.9) that in any category $\mathfrak{K}$ and for a morphism f: $a \to b$ in $\mathfrak{K}$, f is a monomorphism iff the diagram

$$\begin{array}{ccc} a & \xrightarrow{\ \mathrm{id}_a\ } & a \\ {\scriptstyle \mathrm{id}_a}\downarrow & & \downarrow{\scriptstyle f} \\ a & \xrightarrow{\ \ f\ \ } & b \end{array}$$

is a pullback. Thus the desired result follows at once from Proposition 2.10. ■

A monomorphism φ: $\mathfrak{F} \to \mathfrak{G}$ in $\mathfrak{EEns}(\mathfrak{M})$ is called a *regular monomorphism* if the diagram

$$\begin{array}{ccc} \mathbf{f}^*\mathfrak{F}_{\mathrm{Sh}}(\mathbf{Y}) & \xrightarrow{\ \mathfrak{F}_{\#}(\mathbf{f})\ } & \mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) \\ {\scriptstyle \mathbf{f}^*\varphi_{\mathbf{Y}}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathbf{X}}} \\ \mathbf{f}^*\mathfrak{G}_{\mathrm{Sh}}(\mathbf{Y}) & \xrightarrow[\ \mathfrak{G}_{\#}(\mathbf{f})\ ]{} & \mathfrak{G}_{\mathrm{Sh}}(\mathbf{X}) \end{array}$$

is a pullback square for any morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$. Two regular monomorphisms φ: $\mathfrak{F} \to \mathfrak{G}$ and ψ: $\mathfrak{E} \to \mathfrak{G}$ with the same codomain $\mathfrak{G}$ in $\mathfrak{EEns}(\mathfrak{M})$ are said to be *equivalent* if there exists an isomorphism χ: $\mathfrak{F} \to \mathfrak{E}$ in $\mathfrak{EEns}(\mathfrak{M})$ such that $\psi \circ \chi = \varphi$. An equivalence class with respect to this equivalence relation on the regular monomorphisms into $\mathfrak{G}$ is called a *regular subobject* of $\mathfrak{G}$.

Theorem 2.16. For any regular monomorphism φ: $\mathfrak{F} \to \mathfrak{G}$ in $\mathfrak{EEns}(\mathfrak{M})$, there exists a unique morphism χ: $\mathfrak{G} \to \mathfrak{T}$ making the diagram

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{\varphi} & \mathfrak{G} \\ \downarrow & & \downarrow \chi \\ \mathbf{1} & \xrightarrow{\top} & \mathfrak{T} \end{array}$$

a pullback square.

Proof. If such χ exists, then Proposition 2.10 claims that for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, χ_X should be the unique morphism in the topos $\mathfrak{BSh}(\mathbf{X})$ making the diagram

$$\begin{array}{ccc} \mathfrak{F}_{Sh}(\mathbf{X}) & \xrightarrow{\varphi_X} & \mathfrak{G}_{Sh}(\mathbf{X}) \\ \downarrow & & \downarrow \chi_X \\ \mathcal{1}_X & \xrightarrow{\top_X} & \mathcal{T}_X \end{array}$$

a pullback square. Therefore it suffices to show that the family $\{\chi_X\}_{X \in Ob(\mathfrak{M})}$ of morphisms χ_X: $\mathfrak{G}_{Sh}(\mathbf{X}) \to \mathcal{T}_X$ thus chosen for all Boolean locales $\mathbf{X}$ in $\mathfrak{M}$ makes the diagram

$$\begin{array}{ccc} \mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}) & \xrightarrow{\mathbf{f}^*\chi_Y} & \mathbf{f}^*\mathcal{T}_Y \\ \mathfrak{F}_\#(\mathbf{f}) \downarrow & & \downarrow \mathcal{T}^{\mathbf{f}}_\# \\ \mathfrak{F}_{Sh}(\mathbf{X}) & \xrightarrow{\chi_X} & \mathcal{T}_X \end{array}$$

commutative for each morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, which would guarantee that the assignment χ to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of χ_X is the desired morphism in $\mathfrak{E}\mathfrak{E}\mathfrak{ns}(\mathfrak{M})$. To see this, let us consider the diagram

$$\begin{array}{ccc} \mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}) & \xrightarrow{\mathbf{f}^*\varphi_Y} & \mathbf{f}^*\mathfrak{G}_{Sh}(\mathbf{Y}) \\ \mathfrak{F}_\#(\mathbf{f}) \downarrow & & \downarrow \mathfrak{G}_\#(\mathbf{f}) \\ \mathfrak{F}_{Sh}(\mathbf{X}) & \xrightarrow{\varphi_X} & \mathfrak{G}_{Sh}(\mathbf{X}) \\ \downarrow & & \downarrow \chi_X \\ \mathcal{1}_X & \xrightarrow{\top_X} & \mathcal{T}_X \end{array}$$

Since the upper and lower squares are pullback squares, the outer rectangle is also a pullback square by the so-called pullback lemma (cf. MacLane, 1971, Chapter III, §4, Exercise 8). Let us consider also the diagram

$$\begin{array}{ccc}
f^*\mathfrak{F}_{Sh}(Y) & \xrightarrow{f^*\varphi_Y} & f^*\mathfrak{G}_{Sh}(Y) \\
{\scriptstyle f^*\mathfrak{Z}_X}\downarrow & & \downarrow{\scriptstyle f^*\chi_Y} \\
f^*\mathcal{1}_Y & \xrightarrow{f^*\top_Y} & f^*\mathcal{T}_Y \\
\downarrow & & \downarrow{\scriptstyle \mathcal{T}^f_\#} \\
\mathcal{1}_X & \xrightarrow{\top_X} & \mathcal{T}_X
\end{array}$$

Since $\mathbf{f}^*$ preserves pullbacks, the upper square is a pullback square, while Proposition 1.6 claims that the lower square is also a pullback square. Thus the outer rectangle of the above diagram is also a pullback square by the pullback lemma again. Since φ_Y: $\mathfrak{F}_{Sh}(\mathbf{Y}) \to \mathfrak{G}_{Sh}(\mathbf{Y})$ is a monomorphism by Proposition 2.15 and $\mathbf{f}^*$ preserves monomorphisms, the morphism $\mathbf{f}^*\varphi_Y$: $\mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}) \to \mathbf{f}^*\mathfrak{G}_{Sh}(\mathbf{Y})$ is a monomorphism. Therefore the desired commutativity of the second diagram follows. ■

The converse of the above theorem holds.

Theorem 2.17. If the diagram

$$\begin{array}{ccc}
\mathfrak{F} & \xrightarrow{\varphi} & \mathfrak{G} \\
\downarrow & & \downarrow{\scriptstyle \chi} \\
\mathbf{1} & \xrightarrow{\top} & \mathfrak{T}
\end{array}$$

is a pullback square in $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M})$, then the morphism φ: $\mathfrak{F} \to \mathfrak{G}$ is a regular monomorphism.

Proof. Since $\top$ is a monomorphism and the above square is a pullback diagram, φ is a monomorphism (cf. MacLane, 1971, Chapter III, §4, Exercise 5). Therefore it suffices to show that the diagram

$$\begin{array}{ccc}
f^*\mathfrak{F}_{Sh}(Y) & \xrightarrow{\mathfrak{F}_\#(f)} & \mathfrak{F}_{Sh}(X) \\
{\scriptstyle f^*\varphi_Y}\downarrow & & \downarrow{\scriptstyle \varphi_X} \\
f^*\mathfrak{G}_{Sh}(Y) & \xrightarrow[\mathfrak{G}_\#(f)]{} & \mathfrak{G}_{Sh}(X)
\end{array}$$

is a pullback square for each morphism **f**: **X** → **Y** in $\mathfrak{M}$. As in the proof of Theorem 2.16, the outer rectangle of the diagram

$$\begin{array}{ccc}
\mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}) & \xrightarrow{\mathbf{f}^*\varphi_{\mathbf{Y}}} & \mathbf{f}^*\mathfrak{G}_{Sh}(\mathbf{Y}) \\
\downarrow & & \downarrow \mathbf{f}^*\chi_{\mathbf{Y}} \\
\mathbf{f}^*1_{\mathbf{Y}} & \xrightarrow{\mathbf{f}^*\top_{\mathbf{Y}}} & \mathbf{f}^*\mathcal{T}_{\mathbf{Y}} \\
\downarrow & & \downarrow \mathcal{T}^{\mathbf{f}}_{\#} \\
1_{\mathbf{X}} & \xrightarrow{\top_{\mathbf{X}}} & \mathcal{T}_{\mathbf{X}}
\end{array}$$

is a pullback square. This implies that the outer rectangle of the diagram

$$\begin{array}{ccc}
\mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}) & \xrightarrow{\mathbf{f}^*\varphi_{\mathbf{Y}}} & \mathbf{f}^*\mathfrak{G}_{Sh}(\mathbf{Y}) \\
\mathfrak{F}_{\#}(\mathbf{f}) \downarrow & & \downarrow \mathfrak{G}_{\#}(\mathbf{f}) \\
\mathfrak{F}_{Sh}(\mathbf{X}) & \xrightarrow{\varphi_{\mathbf{X}}} & \mathfrak{G}_{Sh}(\mathbf{X}) \\
\downarrow & & \downarrow \chi_{\mathbf{X}} \\
1_{\mathbf{X}} & \xrightarrow{\top_{\mathbf{X}}} & \mathcal{T}_{\mathbf{X}}
\end{array}$$

is a pullback square, for the diagram

$$\begin{array}{ccc}
\mathbf{f}^*\mathfrak{F}_{Sh}(\mathbf{Y}) & \xrightarrow{\mathbf{f}^*\chi_{\mathbf{Y}}} & \mathbf{f}^*\mathcal{T}_{\mathbf{Y}} \\
\mathfrak{F}_{\#}(\mathbf{f}) \downarrow & & \downarrow \mathcal{T}^{\mathbf{f}}_{\#} \\
\mathfrak{F}_{Sh}(\mathbf{X}) & \xrightarrow{\chi_{\mathbf{X}}} & \mathcal{T}_{\mathbf{X}}
\end{array}$$

is commutative, so that the outer rectangles of the second and third diagrams are the same. Since the lower square of the third diagram is also a pullback square, its upper square should be a pullback square by the pullback lemma, which is the desired result. ■

By Theorems 2.16 and 2.17 we can see easily that within the category $\mathfrak{E}\mathfrak{E}\mathfrak{n}\mathfrak{s}(\mathfrak{M})$ the empirical set $\mathfrak{T}$ plays a role of a subobject classifier for regular subobjects. We conclude this section by showing that under highly restrictive conditions even exponentials exist within $\mathfrak{EEns}(\mathfrak{M})$.

An empirical set $\mathfrak{F}$ over $\mathfrak{M}$ is called *flat* if for each morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, $\mathfrak{F}_{\#}(\mathbf{f})$: $\mathbf{f}^*\mathfrak{F}_{\mathrm{Sh}}(\mathbf{Y}) \to \mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})$ is an isomorphism.

Theorem 2.18. Given flat empirical sets $\mathfrak{F}$ and $\mathfrak{G}$ over $\mathfrak{M}$, there exist an empirical set $\mathfrak{H}$ over $\mathfrak{M}$ and a morphism φ: $\mathfrak{H} \times \mathfrak{F} \to \mathfrak{G}$ in $\mathfrak{EEns}(\mathfrak{M})$ such that for any empirical set $\mathfrak{E}$ over $\mathfrak{M}$ and any morphism ψ: $\mathfrak{E} \times \mathfrak{F} \to \mathfrak{G}$ there exists a unique morphism $\hat{\psi}$: $\mathfrak{E} \to \mathfrak{H}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathfrak{H}\times\mathfrak{F} & \xrightarrow{\quad\varphi\quad} & \\ {\scriptstyle \hat{\psi}\times \mathrm{id}_{\mathfrak{F}}}\uparrow & & \mathfrak{G} \\ \mathfrak{E}\times\mathfrak{F} & \xrightarrow[\quad\psi\quad]{} & \end{array}$$

Proof. For each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, let $\mathcal{H}_{\mathbf{X}}$ be the exponential of $\mathfrak{G}_{\mathrm{Sh}}(\mathbf{X})$ by $\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})$ in the topos $\mathfrak{BSh}(\mathbf{X})$ and $\varphi_{\mathbf{X}}$ its evaluation arrow $\mathcal{H}_{\mathbf{X}} \times \mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) \to \mathfrak{G}_{\mathrm{Sh}}(\mathbf{X})$. For each morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, let $\mathbf{f}_{\#}$ be the exponential transpose of $\mathfrak{G}_{\#}(\mathbf{f}) \circ (\mathbf{f}^*\varphi_{\mathbf{Y}}) \circ (\mathrm{id}_{\mathbf{f}^*}\mathcal{H}_{\mathbf{Y}} \times \mathfrak{F}_{\#}(\mathbf{f})^{-1})$. It is not difficult to see that there exists a unique empirical set $\mathfrak{H}$ over $\mathfrak{M}$ such that $\mathfrak{H}_{\mathrm{Sh}}(\mathbf{X}) = \mathcal{H}_{\mathbf{X}}$ for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ and $\mathfrak{H}_{\#}(\mathbf{f}) = \mathbf{f}_{\#}$ for each morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$. It is easy to see that the assignment φ to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of $\varphi_{\mathbf{X}}$ is a morphism from $\mathfrak{H} \times \mathfrak{F}$ to $\mathfrak{G}$ in $\mathfrak{EEns}(\mathfrak{M})$. It is obvious that if such $\hat{\psi}$ as depicted in the theorem exists, then $\hat{\psi}_{\mathbf{X}}$ is the exponential transpose of $\psi_{\mathbf{X}}$ in the topos $\mathfrak{BSh}(\mathbf{X})$ for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$. Therefore it remains to show that the family $\{\hat{\psi}_{\mathbf{X}}\}_{\mathbf{X}\in Ob(\mathfrak{M})}$ so chosen makes the diagram

$$\begin{array}{ccc} \mathbf{f}^*\mathfrak{E}_{\mathrm{Sh}}(\mathbf{Y}) & \xrightarrow{\mathfrak{E}_{\#}(\mathbf{f})} & \mathfrak{E}_{\mathrm{Sh}}(\mathbf{X}) \\ {\scriptstyle \mathbf{f}^*\hat{\psi}_{\mathbf{Y}}}\downarrow & & \downarrow{\scriptstyle \hat{\psi}_{\mathbf{X}}} \\ \mathbf{f}^*\mathfrak{H}_{\mathrm{Sh}}(\mathbf{Y}) & \xrightarrow[\mathfrak{H}_{\#}(\mathbf{f})]{} & \mathfrak{H}_{\mathrm{Sh}}(\mathbf{X}) \end{array}$$

commutative for each morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$. To this end, since $\mathfrak{H}_{\mathrm{Sh}}(\mathbf{X})$ is the exponential of $\mathfrak{G}_{\mathrm{Sh}}(\mathbf{X})$ by $\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})$ and $\varphi_{\mathbf{X}}$: $\mathfrak{H}_{\mathrm{Sh}}(\mathbf{X}) \times \mathfrak{F}_{\mathrm{Sh}}(\mathbf{X}) \to \mathfrak{G}_{\mathrm{Sh}}(\mathbf{X})$ is the evaluation arrow in the topos $\mathfrak{BSh}(\mathbf{X})$, and since the morphism $\mathrm{id}_{\mathbf{f}^*}$ $\mathfrak{E}_{\mathrm{Sh}}(\mathbf{Y}) \times \mathfrak{F}_{\#}(\mathbf{f})$ is an isomorphism, it suffices to show that

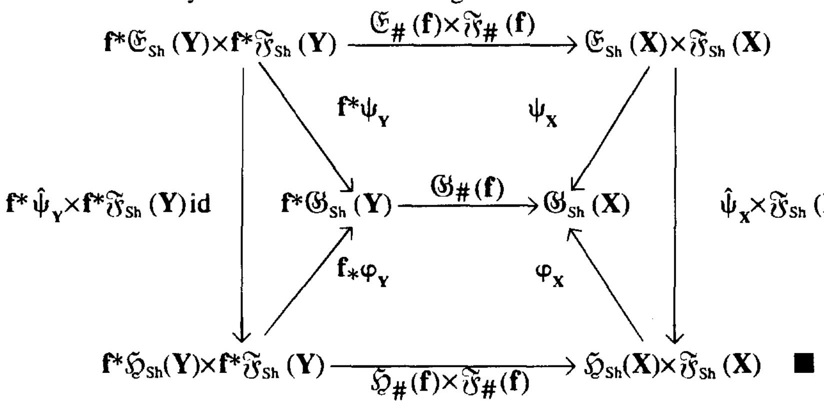
$$\varphi_{\mathbf{X}} \circ ((\hat{\psi}_{\mathbf{X}} \circ \mathfrak{E}_{\#ns}(\mathbf{f}))) \times \mathrm{id}_{\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})}) \circ (\mathrm{id}_{\mathbf{f}^*\mathfrak{E}_{\mathrm{Sh}}(\mathbf{Y})} \times \mathfrak{F}_{\#}(\mathbf{f}))$$

$$= \varphi_{\mathbf{X}} \circ ((\mathfrak{H}_{\#}(\mathbf{f}) \circ \mathbf{f}^*\hat{\psi}_{\mathbf{Y}}) \times \mathrm{id}_{\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})}) \circ (\mathrm{id}_{\mathbf{f}^*\mathfrak{E}_{\mathrm{Sh}}(\mathbf{Y})} \times \mathfrak{F}_{\#}(\mathbf{f}))$$

which is demonstrated as follows:

$$
\begin{aligned}
&\varphi_{\mathbf{X}} \circ ((\hat{\psi}_{\mathbf{X}} \circ \mathfrak{E}_{\#}(\mathbf{f}))) \times \mathrm{id}_{\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})}) \circ (\mathrm{id}_{\mathbf{f}^{*}\mathfrak{E}_{\mathrm{Sh}}(\mathbf{Y})} \times \mathfrak{F}_{\#}(\mathbf{f}))\\
&= \varphi_{\mathbf{X}} \circ (\hat{\psi}_{\mathbf{X}} \times \mathrm{id}_{\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})}) \circ (\mathfrak{E}_{\#}(\mathbf{f}) \times \mathfrak{F}_{\#}(\mathbf{f}))\\
&= \psi_{\mathbf{X}} \circ (\mathfrak{E}_{\#}(\mathbf{f}) \times \mathfrak{F}_{\#}(\mathbf{f}))\\
&= \mathfrak{G}_{\#}(\mathbf{f}) \circ \mathbf{f}^{*}\psi_{\mathbf{Y}}\\
&= \mathfrak{G}_{\#}(\mathbf{f}) \circ \mathbf{f}^{*}\varphi_{\mathbf{Y}} \circ (\mathbf{f}^{*}\hat{\psi}_{\mathbf{Y}} \times \mathrm{id}_{\mathbf{f}^{*}\mathfrak{F}_{\mathrm{Sh}}(\mathbf{Y})})\\
&= \varphi_{\mathbf{X}} \circ (\mathfrak{H}_{\#}(\mathbf{f}) \times \mathfrak{F}_{\#}(\mathbf{f})) \circ (\mathbf{f}^{*}\hat{\psi}_{\mathbf{Y}} \times \mathrm{id}_{\mathbf{f}^{*}\mathfrak{F}_{\mathrm{Sh}}(\mathbf{Y})})\\
&= \varphi_{\mathbf{X}} \circ ((\mathfrak{H}_{\#}(\mathbf{f}) \circ \mathbf{f}^{*}\hat{\psi}_{\mathbf{Y}}) \times \mathrm{id}_{\mathfrak{F}_{\mathrm{Sh}}(\mathbf{X})}) \circ (\mathrm{id}_{\mathbf{f}^{*}\mathfrak{E}_{\mathrm{Sh}}(\mathbf{Y})} \times \mathfrak{F}_{\#}(\mathbf{f}))
\end{aligned}
$$

This calculation was an arrow-chasing in the following diagram, where the above calculation has derived the commutativity of the outer rectangle from the commutativity of all the smaller diagrams:



■

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